

Imputation and Estimation under Nonignorable Nonresponse in Household Surveys with Missing Covariate Information

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In this article we develop and apply new methods for handling not missing at random (NMAR) nonresponse. We assume a model for the outcome variable under complete response and a model for the response probability, which is allowed to depend on the outcome and auxiliary variables. The two models define the model holding for the outcomes observed for the responding units, which can be tested. Our methods utilize information on the population totals of some or all of the auxiliary variables in the two models, but we do not require that the auxiliary variables are observed for the nonresponding units. We develop an algorithm for estimating the parameters governing the two models and show how to estimate the distributions of the missing covariates and the outcomes. The latter distributions are used for imputing the missing values of the nonresponding units and for estimating population means and the variances of the estimators. We consider several test statistics for testing the combined model fitted to the observed data, which enables validating the models used. The new developments are illustrated using a real data set collected as part of the Household Expenditure Survey carried out by the Israel Central Bureau of Statistics in 2005.

Key words: Bootstrap; calibration; Horvitz-Thompson type estimator; nonrespondents' distribution; respondents' distribution.

1. Introduction

Most of the methods dealing with nonresponse assume either explicitly or implicitly that the missing values are “missing at random” (MAR), and that the auxiliary (explanatory) variables are observed for both the respondents and the nonrespondents. These assumptions, however, are not always met in practice. In this article we consider the often practical situation where the probability to respond depends on the outcome value, and possibly also on explanatory variables. For example, the probability of observing income may depend on the income level and socio-demographic variables. For this kind of response mechanism, the missing outcome values are not missing at random (NMAR), since for the nonrespondents the probability of nonresponse depends on the missing outcomes. We consider mostly the case of “unit nonresponse,” where the auxiliary

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(covariate) information for the nonrespondents is likewise unobserved, except for the population totals of some or all of these variables. The covariates' totals are often available from administrative or census records.

We propose a new approach for handling NMAR nonresponse, which does not require knowledge of the covariates for the nonrespondents. We assume a model for the outcome variable under complete response (the "sample model") and a model for the response probability, which is allowed to depend on the outcome and auxiliary variables. Combining the two models yields the "respondents' model," which defines the likelihood for the observed outcomes. In order to utilize the additional information provided by the population totals of the covariates, we add calibration constraints, which match pseudo probability weighted estimates of the totals of the covariates with their known population values. The weights used for these estimates are the products of the sampling weight (inverse of the sample inclusion probability) and the inverse of the response probability under the model. The unknown model parameters are then estimated by an iterative algorithm, which maximizes the likelihood with respect to the parameters governing the sample model and solves the calibration constraints with respect to the parameters of the response probabilities. We prove the convergence of the algorithm and discuss the properties of the resulting estimators.

Having estimated the model parameters, we predict the population mean of the outcome values by use of Horvitz-Thompson (H-T) (Horvitz and Thompson 1952) type estimators, utilizing the estimated response probabilities. Alternatively, when the covariates are observed for all the sampled units, we can estimate the conditional distribution of the outcome values for the nonresponding units given their respective covariates, and then use this distribution for imputing the missing outcomes. Combining the observed and imputed values produces another predictor of the outcome population mean. In the case of missing covariate information for the nonresponding units, the missing values of the covariates are imputed as well using an approximation to their distribution. The variances of the proposed estimators are estimated by parametric and resampling methods. Finally, we test the combined model fitted for the responding units by using standard tests that compare the cumulative hypothesized distribution with the corresponding empirical distribution, and by testing moments of the hypothesized model.

The various procedures considered in this article are illustrated using data collected as part of the Household Expenditure Survey (HES) carried out by the Israel Central Bureau of Statistics in 2005. The initial response rate in this survey was 43%, but after many recalls, the final response rate increased to 90%. This survey provides therefore a rare opportunity of comparing the imputed values after the first interview with the actual values obtained from the recalls.

2. Existing Approaches to Deal With NMAR Nonresponse

In this section we review briefly some of the approaches proposed in the literature to deal with NMAR nonresponse. Let y_i denote the value of an outcome variable Y associated with unit i belonging to a sample $S = \{1, \dots, n\}$, drawn from a finite population $U = \{1, \dots, N\}$ by probability sampling with known inclusion probabilities

$\pi_i = \Pr(i \in s)$. Let $x_i = (x_{1i}, \dots, x_{pi})$ denote the values of p auxiliary variables (covariates) associated with unit i . In what follows we assume that the population outcomes are independent realizations from distributions with probability density functions (*pdf*), $f_P(y_i|x_i; \theta)$, governed by an unknown vector parameter θ . Let $R = \{1, \dots, r\}$ define the sample of respondents with observed outcomes and covariates, and $R^c = \{r+1, \dots, n\}$ define the subsample of nonrespondents, for which the outcomes and possibly the covariates are missing. The response process is assumed to be independent between units.

In the present study we assume that the sampling process is noninformative such that under complete response, $f_S(y_i|x_i) = f(y_i|x_i, i \in S) = f_P(y_i|x_i) \forall i$. Most of the approaches considered in the literature to deal with nonresponse assume (sometimes implicitly) that the missing data are “missing at random” (MAR) (Rubin 1976; Little 1982). This type of nonresponse requires that the probability of responding does not depend on the unobserved data after conditioning on the observed data. Under this condition, if the parameters governing the distribution under full response are distinct from the parameters governing the response process, the nonresponse can be ignored for likelihood and Bayesian inference. Notice that in this case,

$$f_R(y_i|x_i) = f(y_i|x_i, i \in R) = f_S(y_i|x_i) \quad (1)$$

where $f_R(y_i|x_i)$ defines the *marginal pdf* for responding unit i and $f_S(y_i|x_i)$ is the corresponding sample *pdf* defined above. There are many approaches for handling MAR nonresponse – see the books by Schafer (1997) and Little and Rubin (2002), and the recent article by Qin et al. (2008), for comprehensive accounts.

In this article we consider situations where the probability to respond may depend on the outcome value even after conditioning on the covariates. For example, the probability to observe income may depend on the income level as well as on socio-demographic variables. For this kind of response mechanism, the missing outcomes are not missing at random (NMAR).

Suppose first that all the covariates are known for every sampled unit. Define by R_i the response indicator such that $R_i = 1(0)$ if sampled unit i responds (does not respond) to the outcome variable. A possible way to deal with the nonresponse in this situation is by postulating a parametric model for the joint distribution of Y_i and R_i , given x_i . Little and Rubin (2002) distinguish between two ways of formulating the likelihood in this case.

A. Selection Models specify

$$f(y_i, R_i|x_i; \theta, \gamma) = \Pr(R_i|y_i, x_i; \gamma) f_S(y_i|x_i; \theta) \quad (2)$$

where $f_S(y_i|x_i; \theta)$ defines the sample *pdf* (model), $\Pr(R_i|y_i, x_i; \gamma)$ models the response probability and θ and γ denote the (distinct) parameters of the two models. Assuming that the outcomes are independent given the covariates, the *full likelihood* takes in this case the form,

$$L = \prod_{i=1}^r \Pr(R_i = 1|y_i, x_i; \gamma) f_S(y_i|x_i; \theta) \prod_{i=r+1}^n \Pr(R_i = 0|x_i; \theta, \gamma) \quad (3)$$

where $\Pr(R_i = 0|x_i; \theta, \gamma) = 1 - \int \Pr(R_i = 1|y_i, x_i; \gamma)f_S(y_i|x_i; \theta)dy_i$. The response probability is often modeled as

$$\Pr(R_i = 1|y_i, x_i; \gamma) = g(\gamma_0 + \gamma_1^t x_i + \gamma_2 y_i) \quad (4)$$

for some function g taking values in the range $(0,1)$ (see below).

Suppressing for convenience the parameters from the notation, the missing sample values can be imputed in this case by the expectations $E_{R^c}(Y_i|x_i) = E(Y_i|x_i, R_i = 0)$, which can be calculated using Bayes theorem as

$$E_{R^c}(Y_i|x_i) = \int_{-\infty}^{\infty} y f(y_i|x_i, R_i = 0) dy_i = \int_{-\infty}^{\infty} y_i \frac{P(R_i = 0|y_i, x_i) f_S(y_i|x_i)}{\int_{-\infty}^{\infty} P(R_i = 0|y_i, x_i) f_S(y_i|x_i) dy_i} dy_i \quad (5)$$

In practice, the probabilities and densities in (5) are replaced by their estimates as obtained by estimating the unknown parameters. Alternatively, the imputed values can be obtained by drawing at random from the *pdf* $f_{R^c}(y_i|x_i) = f(y_i|x_i, R_i = 0)$, thus accounting for the variability of the outcomes around their expectations. An example of the use of selection models is considered by Greenlees et al. (1982). They assume that the sample model is normal and the probability of responding is logistic.

Selection models allow estimating all the unknown model parameters, but as noted by Little (1994), the use of the likelihood in (3) is based inevitably on strong distributional assumptions. Beaumont (2000) proposes robustifying the model considered by Greenlees et al. (1982) by dropping the normality assumption for the regression residuals. The author estimates the parameters γ by maximizing the likelihood $L = \prod_{i=1}^r P(R_i = 1|y_i, x_i; \gamma) \prod_{i=r+1}^n P(R_i = 0|x_i; \theta, \gamma)$ with respect to γ , assuming that θ is “known,” and the parameters θ by solving weighted least square equations, assuming that γ is “known.” The procedure is carried out iteratively, with the “known” values on a given iteration defined by the estimates obtained on the previous iteration, and with the weights defined by the inverse response probabilities as computed on the previous iteration. A drawback of this method is that the probability $P(R_i = 0|x_i; \theta, \gamma)$ cannot actually be calculated, since the sample *pdf* of $Y_i|x_i$ is not specified. The author deals with this problem by expanding $P(R_i = 1|y_i, x_i, \gamma)$ around the mean $E_S(Y_i|x_i)$, but this amounts to assuming a MAR nonresponse.

B. *Pattern-mixture* models specify

$$f(y_i, R_i|x_i; \psi_m^{(R_i)}, \psi_r) = f(y_i|x_i, R_i; \psi_m^{(R_i)}) \Pr(R_i|x_i; \psi_r) \quad (6)$$

where $f(y_i|x_i, R_i; \psi_m^{(R_i)})$ defines the *pdf* of Y under the different patterns of the missing data, ($R_i = 0, R_i = 1$), $\Pr(R_i|x_i; \psi_r)$ models the response probability given the covariates and $\psi_m^{(R_i)}$ and ψ_r are fixed (unknown) parameters. The likelihood takes in this case the form

$$L = \prod_{i=1}^r f(y_i|x_i, R_i = 1; \psi_m^{(1)}) \Pr(R_i = 1|x_i; \psi_r) \prod_{i=r+1}^n \Pr(R_i = 0|x_i; \psi_r) \quad (7)$$

A major drawback of pattern-mixture models is that the model $f(y_i|x_i, R_i = 0; \psi_m^{(0)})$ holding for the nonrespondents cannot be extracted from the models $f(y_i|x_i, R_i = 1; \psi_m^{(1)})$

and $\Pr(R_i|x_i; \psi_r)$ fitted under this approach, and hence it is not clear how to impute the missing outcomes unless under strong assumptions, which could be hard to test. Little (1993; 1994) discusses plausible relationships between the parameters governing the models holding for the respondents and the nonrespondents and provides examples of the application of selection and pattern-mixture models. Rubin (1987) discusses selection and pattern-mixture models from a Bayesian perspective.

Tang et al. (2003) propose a “pseudo-likelihood” method that uses the *pdf* $g_S(x_i|y_i)$ for the respondents. Application of this method requires specification of the sample *pdf*, $f_S(y_i|x_i)$, and the marginal *pdf*, $g_S(x_i)$, with the later *pdf* possibly replaced by the empirical sample distribution. The method does not require a parametric model for the response probability but it is assumed that it depends only on the outcome. The likelihood takes now the form,

$$L = \prod_{i=1}^r g_S(x_i|y_i; \theta, \eta) = \prod_{i=1}^r \frac{f_S(y_i|x_i; \theta)g_S(x_i; \eta)}{\int f_S(y_i|x; \theta)g_S(x; \eta)dx} \quad (8)$$

Note that although the product is only over the responding units, estimation of the *pdf* $g_S(x_i)$ requires that the covariates are known for all the sample units. The authors point out that the method is robust to misspecification of the response process but the use of this approach does not allow deriving the *pdf* $f_{R^c}(y_i|x_i)$ required for imputing the missing outcomes.

So far we considered methods that assume that the covariates are observed for all the sampled units. Qin et al. (2002) propose a method for the case where the covariates are only known for the respondents. The method assumes a parametric model for $\Pr(R_i = 1|x_i, y_i)$ and known population means, \bar{X}^{pop} , of the covariates. The authors use the empirical likelihood of (y_i, x_i, R_i) ,

$$L = \prod_{i=1}^r \Pr(R_i = 1|y_i, x_i; \gamma)p_i(1 - \lambda)^{n-r} \quad (9)$$

where $\lambda = \Pr(R_i = 1)$ and $p_i = dF_S(y_i, x_i)$ is the “jump” of the joint cumulative distribution $F_S(y_i, x_i)$ at (y_i, x_i) , $i = 1, \dots, r$. The empirical likelihood is maximized under the constraints,

$$\sum_{i=1}^r p_i[\Pr(R_i = 1|y_i, x_i; \gamma) - \lambda] = 0, \sum_{i=1}^r p_i(x_i - \bar{X}^{pop}) = 0; p_i \geq 0, \sum_{i=1}^r p_i = 1 \quad (10)$$

The use of this method addresses the problem of missing covariate information by using the unconditional response probability $\lambda = \Pr(R_i = 1)$ in the likelihood, and it accounts for the known population means of the covariates. However, our experience so far shows that good performance of the procedure depends on having sufficiently accurate initial values for the response model parameters.

Chang and Kott (2008) propose a method for estimating the response probabilities which uses known totals of calibration variables. They assume a parametric model for the response probabilities that may depend on the outcome value, and estimate the unknown parameters of this model by regressing the totals of the calibration variables against their

H-T estimators. The weights used for the H-T estimators are the product of the sampling weights and the inverse of the response probabilities under the model. See Section 3.4 below for more details. The use of this approach allows estimating population totals of interest, but it does not lend itself to imputation of the missing outcomes, since no model is assumed for the outcome values.

3. The Respondents' Distribution and Parameter Estimation

3.1. The Respondents' Distribution and Its Relation to the Sample Distribution

In what follows we denote by x_i the covariates contained in the population model and by v_i the covariates contained in the response model. Let $z_i = (x_i \cup v_i)$.

The *marginal pdf* of the outcome for a responding unit is obtained, similarly to Pfeffermann et al. (1998), as

$$f_R(y_i|z_i) = f(y_i|z_i, i \in S, R_i = 1) = \frac{\Pr(R_i = 1|y_i, v_i, i \in S)}{\Pr(R_i = 1|z_i, i \in S)} f_S(y_i|x_i) \quad (11)$$

where $\Pr(R_i = 1|z_i, i \in S) = \int \Pr(R_i = 1|y_i, v_i, i \in S) f_S(y_i|x_i) dy_i$ and $f_S(y_i|x_i)$ is the sample *pdf* under complete response. As noted before, in this article we assume that the sample *pdf* and the population *pdf* are the same.

Remark 1 As with selection models (Section 2), the use of the respondents' model requires modeling the sample *pdf*, $f_S(y_i|x_i)$ and the response probability, $\Pr(R_i = 1|y_i, v_i, i \in S)$. Notice, however, that the resulting respondents' model can be tested, since it relates to the data observed for the responding units (see Section 6).

By (11), if the sample outcomes and the response are independent between the units, the respondents' likelihood has the form,

$$\begin{aligned} L_{\text{Resp}} &= \prod_{i=1}^r f(y_i|z_i, i \in S, R_i = 1, i \in S; \theta, \gamma) \\ &= \prod_{i=1}^r \frac{\Pr(R_i = 1|y_i, v_i, i \in S; \gamma) f_S(y_i|x_i; \theta)}{\Pr(R_i = 1|z_i, i \in S; \theta, \gamma)} \end{aligned} \quad (12)$$

The notable property of (12) is that it does not require knowledge of the covariates for the nonresponding units, or modeling the distribution of the sampled covariates. As shown later, estimation of the unknown parameters (γ, θ) permits imputing the missing outcomes and covariates and predicting finite population means.

3.2. The Respondents' Likelihood for Generalized Linear Sample Models (GLM)

The GLM is defined as

$$f_S(y_i|x_i; \beta, \phi) = \exp \left\{ a(\phi) \left[y_i \sum_{k=0}^p \beta_k x_{ki} - g \left(\sum_{k=0}^p \beta_k x_{ki} \right) + d(y_i) \right] + \eta(\phi, y_i) \right\} \quad (13)$$

where $x_{k0} = 1$, $\theta' = (\beta', \phi)$ defines the set of unknown parameters and $g(\cdot)$, $a(\cdot)$, $d(\cdot)$ and $\eta(\cdot)$ are known real functions with $g(\cdot)$ strictly increasing and differentiable. In what

follows we assume $\eta(\phi, y_i) = \eta(\phi)$. The log of the likelihood (12) in the case of the GLM (13) is

$$l_{\text{Resp}} = a(\phi) \sum_{i=1}^r \left[y_i \sum_{k=0}^p \beta_k x_{ki} - g \left(\sum_{k=0}^p \beta_k x_{ki} \right) + d(y_i) \right] + r\eta(\phi) \\ + \sum_{i=1}^r \log[\Pr(R_i = 1 | y_i, v_i, i \in S; \gamma)] - \sum_{i=1}^r \log[\Pr(R_i = 1 | z_i, i \in S; \beta, \phi, \gamma)] \quad (14)$$

Denote $\pi(y_i, v_i; \gamma) = \Pr(R_i = 1 | y_i, v_i, i \in S; \gamma)$. Taking the derivatives of the log-likelihood (14) with respect to β and ϕ and equating them to zero, we obtain after some tedious algebra the following equations:

$$l_k = \sum_{i=1}^r [y_i - E_R(Y_i | z_i; \beta, \phi, \gamma)] x_{ki} = 0, \quad k = 0, \dots, p \quad (15a)$$

$$l_{p+1} = \sum_{i=1}^r [d(y_i) - E_R(d(Y_i) | z_i; \beta, \phi, \gamma)] = 0 \quad (15b)$$

$$\text{where } E_R(Y_i | z_i; \beta, \phi, \gamma) = \int y_i f_R(y_i | z_i; \beta, \phi, \gamma) dy_i = \int y_i \frac{\pi(y_i, v_i; \gamma) f_S(y_i | x_i; \beta, \phi)}{\int \pi(y_i, v_i; \gamma) f_S(y_i | x_i; \beta, \phi) dy_i} dy_i$$

Let $\gamma = (\gamma_0, \dots, \gamma_{q+1})$. Taking derivatives of the log-likelihood (14) with respect to γ and assuming that the order of integration and differentiation can be interchanged, we obtain after equating the derivatives to zero

$$\sum_{i=1}^r E_R \left(\frac{\partial \pi(y_i, v_i; \gamma)}{\partial \gamma_k} \frac{1}{\pi(y_i, v_i; \gamma)} \middle| z_i \right) = \sum_{i=1}^r \left(\frac{\partial \pi(y_i, v_i; \gamma)}{\partial \gamma_k} \frac{1}{\pi(y_i, v_i; \gamma)} \right), \quad (16) \\ k = 0, \dots, (q+1)$$

For example, if the response probability is logistic such that

$$\pi(y_i, v_i; \gamma) = \left\{ 1 + \exp \left[- \left(\gamma_0 + \sum_{k=1}^q v_{ki} \gamma_k + y_i \gamma_{q+1} \right) \right] \right\}^{-1} \\ = \{ 1 + \exp [-(1, v'_i, y_i) \gamma] \}^{-1}, \text{ the equations are:}$$

$$\sum_{i=1}^r E_R(\pi(y_i, v_i; \gamma) | z_i) = \sum_{i=1}^r \pi(y_i, v_i; \gamma); \quad (17) \\ \sum_{i=1}^r v_{1i} E_R(\pi(y_i, v_i; \gamma) | z_i) = \sum_{i=1}^r v_{1i} \pi(y_i, v_i; \gamma) \\ \vdots \\ \sum_{i=1}^r v_{qi} E_R(\pi(y_i, v_i; \gamma) | z_i) = \sum_{i=1}^r v_{qi} \pi(y_i, v_i; \gamma); \\ \sum_{i=1}^r E_R(y_i \pi(y_i, v_i; \gamma) | z_i) = \sum_{i=1}^r y_i \pi(y_i, v_i; \gamma)$$

The solution of Equations (15a), (15b) and (16) (or 17 in the logistic case) yields the maximum likelihood estimators (MLEs) for (β', ϕ, γ') .

3.3. Calibration Constraints

Our experience so far shows that maximization of the likelihood (12) ((14) in the GLM case) may be unstable. In what follows we assume knowledge of the population totals of all the covariates included in the response model and at least one of the covariates included in the sample model. We later relax this requirement. We also assume that the sampling weights, $\{w_i = (1/\pi_i) = 1/\Pr(i \in S)\}$, are known for every responding unit in the sample. The additional information contained in the population totals is not part of the likelihood (14). We utilize this information by imposing the following constraints. Let the sample *pdf* be the GLM defined by (13) and denote by $Z^{pop} = (Z_1^{pop}, \dots, Z_{t^*}^{pop}) = (V_1^{pop}, \dots, V_q^{pop}, X_1^{pop}, \dots, X_{p^*}^{pop})$ the known population totals, where $p^* \leq p = \dim(x_i)$, $t^* \leq t = \dim(z_i)$. The calibration constraints are

$$\sum_{i=1}^r w_i \frac{v_{ki}}{\pi(y_i, v_i; \gamma)} = V_k^{pop}, k = 1, \dots, q; \quad \sum_{i=1}^r w_i \frac{1}{\pi(y_i, v_i; \gamma)} = N \quad (18a)$$

When the response model has an intercept, we use the additional constraint

$$\sum_{i=1}^r w_i \frac{\tilde{\beta}' \tilde{x}_i}{\pi(y_i, v_i; \gamma)} = \tilde{\beta}' \tilde{X}^{pop} \quad (18b)$$

where $\tilde{x}_i = (x_{1i}, \dots, x_{p^*i})$, $\tilde{X}^{pop} = (X_1^{pop}, \dots, X_{p^*}^{pop})$ and $\tilde{\beta}$ is the vector of coefficients of \tilde{x}_i in the sample model. Notice that if $E_S(Y_i|x_i; \beta, \phi) = \sum_{k=0}^p \beta_k x_{ki}$ (the sample expectation is linear) and $p = p^*$, the constraint (18b) implies,

$$\sum_{i=1}^r w_i \frac{E_S(Y_i|x_i)}{\pi(y_i, v_i; \gamma)} = \sum_{j=1}^N E_S(Y_j|x_j) = \sum_{j=1}^N E_p(Y_j|x_j)$$

since we assume that the population and sample models are the same.

Remark 2 The left-hand sides of (18a) and (18b) are the familiar H-T estimators of the corresponding totals under the following two-phase sampling process: in the first phase a sample S of size n is sampled with inclusion probabilities $\pi_i = 1/w_i$; in the second phase the sampled units respond with probabilities $\pi(y_i, v_i; \gamma)$ (Särndal and Swensson 1987).

3.4. Estimation Algorithm, Properties of Estimators

In order to utilize the additional information provided by knowledge of the population totals, we replace Equation (16) by Equations (18a) and (18b), and use the following iterative algorithm.

Let $(\beta^{(0)}, \phi^{(0)})$ denote initial values for (β, ϕ) governing the sample *pdf* $f_s(y_i|x_i; \beta, \phi)$.

Step j: For given $(\hat{\beta}^{(j)}, \hat{\phi}^{(j)})$ from iteration j , set $(\beta, \phi) = (\hat{\beta}^{(j)}, \hat{\phi}^{(j)})$ and solve Equations (18a) and (18b) as a function of the unknown parameters γ governing the response probabilities $\pi(y_i, v_i; \gamma)$. This step yields new estimators $\hat{\gamma}^{(j+1)}$.

Step (j + 1): Solve (15a) and (15b) with respect to (β, ϕ) , with γ equal to $\gamma^{(j+1)}$. This step yields new estimators $(\hat{\beta}^{(j+1)}, \hat{\phi}^{(j+1)})$. Continue the iterations until convergence.

Our experience shows that the use of this algorithm simplifies the computation of the estimators and is much more stable than the solution of the likelihood equations (15a), (15b) and (16). It also utilizes the additional information provided by the known totals of the covariates. The solution of Equations (15a) and (15b) for fixed γ is outlined in Appendix A.

Let $l(\theta, \gamma) = \partial \log(L_{\text{Resp}})/\partial \theta$, with L_{Resp} defined by (12). Denote by $h(\theta, \gamma)$ the differences between the left- and right-hand side of the system of Equations (18a) – (18b), which can be written then as $h(\theta, \gamma) = 0$, and let $\hat{\xi}^l = (\hat{\theta}^l, \hat{\gamma}^l)$ define the estimator obtained by application of the algorithm.

Theorem 1 Suppose that:

- I) The population (sample) model belongs to the family of generalized linear models.
- II) $0 < \pi(y_i, v_i; \gamma) < 1$, with bounded first derivatives with respect to γ .
- III) The functions $l(\theta, \gamma)$ and $h(\theta, \gamma)$ are continuous and twice differentiable with respect to (θ, γ) in a compact neighborhood of the solution $\xi'_0 = (\theta'_0, \gamma'_0)$ of Equations 15(a) – 15(b) and 18(a) – 18(b).
- IV) The matrices $\partial l(\theta, \gamma)/\partial \theta$ $\partial h(\theta, \gamma)/\partial \gamma$ are nonsingular in a neighborhood of the true vector parameter $\tilde{\xi} = (\tilde{\theta}^l, \tilde{\gamma}^l)'$.
- V) $\tilde{\xi}$ is the unique solution of the equation $E[U(\tilde{\xi})] = 0$, where $\xi^l = (\theta^l, \gamma^l) = (\theta_0, \dots, \theta_{p+1}, \gamma_0, \dots, \gamma_{q+1})$, $U = (l', h')' = (l_0, \dots, l_{p+1}, h_0, \dots, h_{q+1})'$ and the expectation is taken over all possible samples of respondents and all possible sample outcomes under the sample distribution.

Then, as $N \rightarrow \infty$, $n \rightarrow \infty$ such that $(N/n) < \infty$, the estimator $\hat{\xi}^l = (\hat{\theta}^l, \hat{\gamma}^l)$ converges in probability to the solution $\xi'_0 = (\theta'_0, \gamma'_0)$. The theorem is proved in Appendix B.

Next we establish the consistency and asymptotic normality of the estimator $\hat{\xi} = (\hat{\theta}^l, \hat{\gamma}^l)'$. For this, note first that Equations (15a) – (15b) and (18a) – (18b) can be written as $1/n \sum_{i=1}^n \varphi(R_i, y_i, z_i; \theta, \gamma) = 0$, where as before, $\theta^l = (\beta^l, \phi)$ and R_i is the response indicator. (For $R_i = 0$, $\varphi(R_i, y_i, z_i; \theta, \gamma) = 0$). In the theorem below all the expectations are taken over all possible samples of respondents and all possible outcomes under the sample distribution.

Theorem 2 Suppose that:

- (i) The true vector parameter $\tilde{\xi} = (\tilde{\theta}^l, \tilde{\gamma}^l)'$ is an interior point of the parameter space,
 - (ii) $\varphi(R_i, y_i, z_i; \xi)$ is continuously differentiable in a neighborhood δ of $\tilde{\xi}$,
 - (iii) $E[\varphi(R_i, y_i, z_i, \tilde{\xi})] = 0$ and $\tilde{\xi}$ is the unique solution of the equations $E[U(\xi)] = 0$ where $\xi^l = (\theta^l, \gamma^l) = (\theta_0, \dots, \theta_{p+1}, \gamma_0, \dots, \gamma_{q+1})$ and $U = (l', h')' = (l_0, \dots, l_{p+1}, h_0, \dots, h_{q+1})'$.
 - (iv) $E[\varphi(R_i, y_i, z_i, \tilde{\xi})\varphi(R_i, y_i, z_i, \tilde{\xi})'] < \infty$,
 - (v) $E[\sup_{\xi \in \delta} \|\partial \varphi(y_i, z_i, \xi)/\partial \xi\|] < \infty$.
- Then the estimator $\hat{\xi} = (\hat{\theta}^l, \hat{\gamma}^l)'$ is consistent for $\tilde{\xi}$ and $\sqrt{n}(\hat{\xi} - \tilde{\xi}) \xrightarrow{D} N[0, V(\tilde{\xi})]$. The theorem is proved in Appendix C.

3.5. Alternative Algorithm for Estimating the Response Probabilities

Another possibility of utilizing known covariate totals for estimating the parameters γ governing the model for the response probabilities is by applying an approach proposed by Chang and Kott (2008). By this approach, the totals of calibration variables C_1, \dots, C_K , which may contain some or all of the covariates in the response model, are regressed against their H-T estimators. Thus, in the case that the probability of responding depends on the outcome variable and $q + 1$ covariates (including an intercept), the method requires that $K \geq q + 2$. The major difference between the calibration equations (18) and this method is that it allows utilizing more population totals than the totals of the variables included in Z . In particular, population totals of variables not included in the model for the response probabilities may be used. This results in more equations than estimated parameters and hence, potentially, in more stable estimators.

Let c_i denote the values of the calibration variables for unit i . Chang and Kott (2008) estimate the unknown parameters γ by setting the nonlinear regression equations

$$C^{pop} = \sum_{i=1}^r w_i \frac{c_i}{\pi(y_i, v_i; \gamma)} + \varepsilon^* \text{ where } C^{pop} = \sum_{j=1}^N c_j \text{ and } \varepsilon^* \text{ is a vector of errors.}$$

The parameters γ are estimated by applying the iterative algorithm

$$\hat{\gamma}^{(j+1)} = \hat{\gamma}^{(j)} + \{ \hat{H}(\hat{\gamma}^{(j)})^T V^{-1}(\hat{\gamma}^{(j)}) \hat{H}(\hat{\gamma}^{(j)}) \}^{-1} \hat{H}(\hat{\gamma}^{(j)})^T V^{-1}(\hat{\gamma}^{(j)}) \\ \times \left(C^{pop} - \sum_{i=1}^r w_i \frac{c_i}{\pi(y_i, v_i; \hat{\gamma}^{(j)})} \right) \quad (19)$$

$$\text{where } \hat{H}(\hat{\gamma}^{(j)}) = \frac{\partial \left[\sum_{i=1}^r w_i \frac{c_i}{\pi(y_i, v_i; \gamma)} \right]}{\partial \gamma} \Bigg|_{\gamma = \hat{\gamma}^{(j)}} \text{ and } \hat{V}^{-1}(\hat{\gamma}^{(j)}) \text{ is the inverse of an}$$

estimator for the quasi-randomization variance of $\sum_{i=1}^r w_i \frac{c_i}{\pi(y_i, v_i; \gamma)}$, computed at $\gamma = \hat{\gamma}^{(j)}$

Remark 3 Chang and Kott (2008) do not assume a model for the outcome variable so that their approach is restricted to estimation of the model for the response probabilities, which can be used for estimating finite population totals, but not for imputation. However, the following theorem holds (the proof can be obtained from the authors).

Theorem 3 Let γ^* be the estimator obtained by application of (19), and θ^* be the solution of the equations $l(\theta, \gamma^*) = 0$ (Equations 15(a) – 15(b) with $\gamma = \gamma^*$). Suppose that $0 < \pi(y_i, v_i; \gamma) < 1$ with bounded first derivatives with respect to γ . Then under some added regularity conditions as $n \rightarrow \infty$ $(\theta^*, \gamma^*) \xrightarrow{p} (\tilde{\theta}, \tilde{\gamma})$, and $\sqrt{n}(\theta^{*'}, \gamma^{*'}) \xrightarrow{D} N[(\tilde{\theta}', \tilde{\gamma}'), \Sigma^*]$ for some fixed matrix Σ^* .

Remark 4 An obvious advantage of the use of (19) instead of (18a) and 18(b) for the estimation of (θ, γ) is that it does not require knowledge of the population totals of all the covariates featuring in the model for the response probabilities and at least one of the covariates featuring in the model for the outcome variable. On the other hand, our experience so far shows that the use of 18(a) and 18(b) yields better parameter estimates and better imputations when the totals required for the use of these equations are known.

4. Imputation and Estimation of Population Means

Denote for nonresponding unit i ,

$$\begin{aligned}\hat{f}_{R^c}(y_i|z_i) &= \frac{\Pr(R_i = 0|y_i, v_i, i \in S; \hat{\gamma})f_S(y_i|x_i; \hat{\theta})}{\Pr(R_i = 0|z_i, i \in S; \hat{\theta}, \hat{\gamma})} \\ &= \frac{[1 - \hat{\pi}(y_i, v_i)]f_S(y_i|x_i; \hat{\theta})}{1 - \Pr(R_i = 1|z_i, i \in S; \hat{\theta}, \hat{\gamma})}\end{aligned}\quad (20)$$

$$\hat{\pi}(y_i, v_i) = \pi(y_i, v_i; \hat{\gamma}), \hat{E}_{R^c}(Y_i|z_i) = E_{R^c}(Y_i|z_i; \hat{\theta}, \hat{\gamma}) \quad (21)$$

the estimate of the *pdf* of the outcome and the estimates of the response probability and the expectation of the outcome respectively. The expectation $E_{R^c}(Y_i|z_i)$ is with respect to the *pdf* $f_{R^c}(y_i|z_i)$. The estimates in (20) and (21) provide several possibilities for the imputation of the missing values and the estimation of the population mean of the outcome.

When the covariates for the nonrespondents are unknown, the population mean of the outcome can be estimated using the (pseudo) H-T estimator

$$\hat{Y}_{(1)} = \frac{1}{N} \sum_{i=1}^r w_i y_i / \hat{\pi}(y_i, v_i) \quad (22)$$

When the covariates are known for all the sampled units, another plausible estimator is

$$\hat{Y}_{(2)} = \frac{1}{N} \sum_{i=1}^n w_i y_i^*; y_i^* = y_i \text{ if } i \in R, y_i^* = y_i^{imp} \text{ if } i \in R^c \quad (23)$$

The imputed values can be computed either as

$$y_i^{imp} = \hat{E}_{R^c}(Y_i|z_i) \quad (24)$$

or by generating one or more observations from the *pdf* $\hat{f}_{R^c}(y_i|z_i)$ and taking the average of these observations as the imputed value, using multiple imputation techniques (Rubin 1987; Schafer and Schenker 2000).

Remark 5 It is important to emphasize that no new model is assumed for the outcomes of the nonresponding units. This model is defined mathematically by the relationship (20). The sample model, $f_S(y_i|x_i; \theta)$, and the model for the response probabilities, $\pi(y_i, v_i; \gamma)$, define the model holding for the outcomes of the responding units and this model can be validated by application of goodness of fit test statistics since it refers to the observed data (see Section 6).

The predictor $\hat{Y}_{(2)}$ in (23) requires that the covariates are known for every unit in the sample. When the covariates are only known for the respondents, we can first impute the missing covariates for the nonrespondents from the probability function $P_{Z|0}(z_i) = \Pr(Z_i = z_i | R_i = 0, i \in S)$, and then predict the outcome value as described above. Following Sverchkov and Pfeffermann (2004), the latter probability function can be expressed as

$$P_{Z|0}(z_i) = \frac{\Pr(R_i = 0 | Z_i = z_i, i \in S)}{\Pr(R_i = 0 | i \in S)} \Pr(Z_i = z_i | i \in S) \tag{25}$$

$$= \frac{\Pr(R_i = 0 | Z_i = z_i, i \in S) \Pr(Z_i = z_i | R_i = 1, i \in S) \Pr(R_i = 1 | i \in S)}{\Pr(R_i = 0 | i \in S) \Pr(R_i = 1 | Z_i = z_i, i \in S)}$$

Estimating $\hat{\Pr}(Z_i = z_i | R_i = 1, i \in S) = (1/r)$ and $\hat{\Pr}(R_i = 1 | i \in S) = r / \left[\sum_{j=1}^r [1 / \hat{\pi}(z_j)] \right]$ where $\hat{\pi}(z_i) = \Pr(R_i = 1 | z_i, i \in S; \hat{\theta}, \hat{\gamma})$, the probability $P_{Z|0}(z_i)$ can be estimated as

$$\hat{P}_{Z|0}(z_i) = \frac{[1 - \hat{\pi}(z_i)]}{\hat{\pi}(z_i) \left[\sum_{j=1}^r (1 / \hat{\pi}(z_j)) - r \right]}, \quad z_i \in R \tag{26}$$

Remark 6 The estimator (26) assumes that the covariates in the subsample of the nonrespondents take the same values as in the subsample of the respondents (but with different frequencies). It is actually easy to show that for any vector covariate z^* such that $0 < \Pr(R = 1 | Z = z^*) < 1$, $\Pr(Z = z^* | R = 1) \neq 0 \Leftrightarrow \Pr(Z = z^* | R = 0) \neq 0$. Note also that $\sum_{j=1}^r \hat{P}_{Z|0}(z_j) = 1$. When $\dim(z_i)$ is small, the estimate $\hat{\Pr}(Z_i = z_i | R_i = 1, i \in S) = (1/r)$ can be enhanced by use of more advanced density estimation methods. In what follows we denote the estimator with imputed covariates and outcomes by $\hat{Y}_{(3)}$.

5. Estimation of Variances of Estimators of Population Means

In Section 4 we considered several estimators of the population mean of the outcome variable. In order to estimate the variance of these estimators, we can apply a parametric bootstrap procedure, distinguishing between estimation of the conditional variance given the observed covariates for the respondents (and thus conditioning also on the number of respondents), and the unconditional variance over all possible samples of respondents (and thus also over all possible numbers of respondents). The bootstrap procedure for estimating the conditional variance consists of the following steps:

1. Generate a large number B of samples of outcomes of size r from the estimated respondents' distributions $f_R(y_i | z_i; \hat{\theta}, \hat{\gamma})$ with given (original) covariates z_i .
2. For each new sample, reestimate (θ, γ) and then compute the estimators $\hat{Y}_{(1)} - \hat{Y}_{(3)}$ using the new parameter estimators. Estimate,

$$V\hat{a}r(\hat{Y}_{(k)}) = \frac{1}{B} \sum_{b=1}^B \left(\hat{Y}_{(k)}^{(b)} - \bar{Y}_{(k)}^A \right)^2; \quad \bar{Y}_{(k)}^A = \frac{1}{B} \sum_{b=1}^B \hat{Y}_{(k)}^{(b)}, \quad k = 1, 2, 3 \tag{27}$$

where $\hat{Y}_{(k)}^{(b)}$ denotes the estimator computed from bootstrap sample $b = 1, \dots, B$.

For estimating the unconditional variances we first impute the missing covariates for the nonrespondents, if they are missing. Next we generate the outcomes for the whole sample using the estimated sample distribution, $f_S(y_i|x_i; \hat{\theta})$ and then select respondents with probabilities $\pi(y, v_i; \hat{\gamma})$. In this case the number of respondents and their covariates change from one bootstrap sample to the other. The rest of the computations are the same as for the conditional variances. The whole process is repeated B times.

Another way of estimating the variance of the H-T estimator $\hat{Y}_{(1)}$ is by estimating the conditional variance

$$Var_c(\hat{Y}_{(1)}) = Var \left[\tilde{Y}_{(1)} | \tilde{T}_{v,x} = \left(N, V_1^{pop}, \dots, V_q^{pop}, \hat{\beta}' \tilde{X}^{pop} \right)' \right] \quad (28)$$

where $\tilde{Y}_{(1)} = \frac{1}{N} \sum_{i=1}^r \frac{w_i y_i}{\pi(y_i, v_i; \gamma)}$, and

$$\tilde{T}_{v,x} = \left[\sum_{i=1}^r w_i \frac{1}{\pi(y_i, v_i; \gamma)}, \sum_{i=1}^r w_i \frac{v_{1i}}{\pi(y_i, v_i; \gamma)}, \dots, \sum_{i=1}^r w_i \frac{v_{qi}}{\pi(y_i, v_i; \gamma)}, \sum_{i=1}^r w_i \frac{\hat{\beta}' \tilde{x}_i}{\pi(y_i, v_i; \gamma)} \right]'$$

This variance accounts for the calibration constraints used for estimating the model parameters (Equations 18(a) and 18(b)) and hence the response probabilities. (Deville and Tille (2005) consider a similar variance estimator in a different context.) Denote $\sigma_{11} = Var(\tilde{Y}_{(1)})$, $\Sigma_{22} = Var(\tilde{T}_{v,x})$ and $\sigma'_{12} = Cov(\tilde{Y}_{(1)}, \tilde{T}_{v,x})$. Assuming $\tilde{Y}_{(1)} \cong \delta' \tilde{T}_{v,x} + \varepsilon$, $E(\varepsilon | \tilde{T}_{v,x}) = 0$ for some vector δ , (e.g., assuming asymptotic normality of $(\tilde{Y}_{(1)}, \tilde{T}_{v,x})$),

$$Var_c(\hat{Y}_{(1)}) = \sigma_{11} - \sigma'_{12} \Sigma_{22}^{-1} \sigma_{12} \quad (29)$$

The variance components in (29), and hence the conditional variance of $\hat{Y}_{(1)}$, can be computed and estimated with respect to the randomization distribution over all possible samples of respondents and all possible original samples, or over all possible samples of respondents and all possible outcomes under the sample model, with the unknown model parameters replaced by their original sample estimates. The estimator (29) does not require resampling procedures.

Remark 7 The variance of the estimators $\hat{Y}_{(2)}$ and $\hat{Y}_{(3)}$, which use observed and imputed values could possibly be estimated also using multiple imputation theory (Rubin 1987; Schafer and Schenker 2000). However, empirical results obtained so far indicate that a textbook application of this method in the present context does not produce well-behaved estimators. We are investigating this problem.

6. Testing the Goodness of Fit of the Model

As noted before, the pdf (11), which is fitted for the responding units, can be tested since it refers to the observed data. In principle, one faces the classical problem of having a random sample from a hypothesized pdf which has to be validated. In what follows we consider several goodness of fit test statistics that seem appropriate for our problem.

6.1. Classical Tests

Suppose first that the true model parameters (θ, γ) are known. Denote by $U_i(y) = \int_{-\infty}^y f_R(t|z_i; \theta, \gamma) dt$ the hypothesized cumulative respondent distribution function (*cdf*) of $y_i|z_i$, $i = 1, \dots, r$. For an absolutely continuous *cdf*, the random variables $U_i(\cdot)$ are independent Uniform $[0,1]$ variables since the responses y_i are independent given the covariates. Denote by u_1, \dots, u_r the values of U_1, \dots, U_r at the sample values y_1, \dots, y_r respectively, and let F_{emp} define the empirical distribution of u_1, \dots, u_r . Following Landsman (2008), we apply three classical goodness of fit tests to the ordered values $u_{(1)}, \dots, u_{(r)}$. The tests are:

$$\text{Kolmogorov - Smirnov: } KS = \max_i |F_{Emp}(u_{(i)}) - u_{(i)}| \quad (30)$$

$$\text{Cramer - von Misses: } CM = \frac{1}{12r} + \sum_{i=1}^r \left[u_{(i)} - \frac{2i-1}{2r} \right]^2 \quad (31)$$

Anderson - Darling:

$$AD = -r - \frac{1}{r} \sum_{i=1}^r [(2i-1) \ln(u_{(i)}) + (2r+1-2i) \ln(1-u_{(i)})] \quad (32)$$

So far, we have assumed known parameter values. When the test statistics are computed with estimated parameters, the asymptotic distribution of the three statistics depends in a complex way on the hypothesized model, the true model parameters and the method of estimation. Correct critical values can be obtained in this case by use of parametric bootstrap. The procedure consists of generating a large number of samples from the estimated hypothesized model, reestimating the unknown parameters from each bootstrap sample and then computing the corresponding test statistics. The bootstrap distribution of these statistics provides approximate critical values for the null distribution, with correct order of error. See Babu and Rao (2004) for regularity conditions establishing the use of this procedure.

6.2. Other Tests

In addition to the classical tests considered above, we propose tests that compare the theoretical moments of the fitted distributions with their H-T estimators. In what follows we illustrate the use of these statistics for the case where the population *pdf* is normal, but the tests can be modified to other population distributions.

Under normality of the population *pdf*, $Y_i = x_i' \beta + \varepsilon_i$, $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ and we can test, for example, $H_0^{(3)} : \mu^{(3)} = E(\varepsilon^3) = 0$, or $H_0^{(4)} : \mu^{(4)} = E(\varepsilon^4) = 3\sigma_\varepsilon^4$, using the following test statistics:

$$H_0^{(3)} \rightarrow C_{(3)} = \frac{1}{\sqrt{\hat{V}^{(3)}}} \frac{1}{N} \sum_{i=1}^r w_i \frac{(y_i - x_i' \hat{\beta})^3}{\hat{\sigma}_\varepsilon^3 \pi(y_i, v_i; \hat{\gamma})} \quad (33)$$

$$H_0^{(4)} \rightarrow C_{(4)} = \frac{1}{\sqrt{\hat{V}^{(4)}}} \left(\frac{1}{N} \sum_{i=1}^r w_i \frac{(y_i - x_i' \hat{\beta})^4}{\hat{\sigma}_\varepsilon^4 \pi(y_i, v_i; \hat{\gamma})} - 3 \right) \quad (34)$$

$$\text{where } \hat{V}^{(3)} = \text{Var}_c \left(\frac{1}{N} \sum_{i=1}^r w_i \frac{(y_i - x_i' \hat{\beta})^3}{\pi(Y_i, v_i; \hat{\gamma})} \right) \text{ and } \hat{V}^{(4)} = \text{Var}_c \left(\frac{1}{N} \sum_{i=1}^r w_i \frac{(y_i - x_i' \hat{\beta})^4}{\pi(y_i, v_i; \hat{\gamma})} \right)$$

are the conditional variances given the calibration constraints 18(a)–18(b), similarly to Section 5. Critical values for the test statistics $C_{(3)}$ and $C_{(4)}$ can be obtained by parametric bootstrap, similarly to the procedure described in Section 6.1. Alternatively, for large r one can use a normal approximation by relating to an appropriate central limit theorem.

7. Application of Methods to Household Expenditure Survey

7.1. Study Population and Outcome Variable

In this section we illustrate and study the performance of the proposed approach by using data collected as part of the Household Expenditure Survey (HES) carried out by the Israel Central Bureau of Statistics in 2005. The survey collects information on socio-demographic characteristics and the income of each member of the sampled households (HHs), as well as information on the HH expenditure. The HHs were sampled with equal probabilities by a two-stage sampling design. The initial response rate in this survey was as low as 43%, but after many recalls it increased to 90% of the sampled HHs. In what follows we restrict ourselves to HHs where the head of the HH is an employee, aged 25–64 and born in Israel. We only consider HHs where at least one of its members worked during the three months preceding the interview. After removing four HHs as outliers, the total sample size is $n = 1,717$, with $r = 629$ responding HHs and $n - r = 1,088$ nonresponding HHs, so that for our sample the response rate is 37%. The head of the HH is the member of the family who has the highest income. The target outcome variable is the *household income per standard person*.

For the present study we define the responding HHs to be the HHs that responded on the first interview. The nonresponding HHs are the HHs which did not respond on the first interview but responded on one of the later interviews, such that the data for both the responding and the nonresponding HHs are actually known. This allows comparing the imputed values with the corresponding true values, assuming that the reported incomes are not affected by being collected at a later interview and that the late respondents can be viewed as a random sample from the nonresponding units. (Recall that the eventual response rate is 90%). As mentioned above, the HHs were sampled with equal probabilities and we assume therefore that the population model and the sample model under full response are the same.

7.2. Sample Model and Response Probabilities

We assume (and validate in Section 7.5) that the sample distribution of the outcome (under full response) given the covariates is *lognormal*, and that the response probabilities given the outcome and the covariates can be modeled by the *logistic*

function, that is,

$$y_i = x_i' \beta + \varepsilon_i; \varepsilon_i \sim N(0, \sigma_\varepsilon^2), i \in S \quad (35)$$

$$P(R_i = 1 | y_i, v_i) = [1 + e^{-(\gamma_0 y_i + \gamma_1' v_i)}]^{-1} \quad (36)$$

where y_i is the log(income) per standard person in household i and x_i and v_i are the corresponding vector covariates. The covariates include characteristics of the head of the HH: gender, age, occupation (“Occ.”) and number of years at school (“Sch.”), as well as HH characteristics: number of earners (“earners”), HH size (“HH size”) and district of residence (“Dist.”). Most of the covariates included in the sample model (35), and in particular the outcome variable log(income) are nonsignificant when included in the response model (36). However, when removing the nonsignificant covariates from the model, log(income) is significant and the model contains fewer covariates. We compared the imputations and the variances of the estimators of the population mean income as obtained under the two response models and obtained similar results. In what follows we only consider the reduced model.

Tables 1 and 2 show the estimated coefficients of Models (35) and (36) and their estimated standard errors as obtained when fitting the two models separately to all the sample data (respondents and “nonrespondents”), and when fitting the respondents’ model (11) to only the responding units, using the algorithm described in Section 3.4. (See Appendix C for computation of the standard errors when fitting the respondents’ model.) For the application of the algorithm we took the true population totals of the covariates included in the logistic response model to be the corresponding sample totals.

The figures in the two tables strongly suggest that the coefficients can be estimated sufficiently accurately based only on the model holding for the responding units (the practical scenario). Notice that the standard errors when fitting the models to the entire sample are considerably smaller, which is obvious since the sample size in this case is

Table 1. Sample model coefficients and standard errors when fitting the model to all the sampled HHs (Respondents and Nonrespondents), and when fitting the model to only the responding HHs

| | Const. | Gender | Age | Dist. 21 | Dist. 41 | Dist. 42 | Dist. 43 | Dist. 44 |
|-----------|---------|--------|--------|----------|----------|----------|----------|----------|
| All HHs | 7.29 | -0.12 | 0.02 | -0.18 | 0.16 | 0.13 | 0.19 | 0.18 |
| Std error | 0.071 | 0.023 | 0.001 | 0.060 | 0.044 | 0.033 | 0.045 | 0.034 |
| Respond. | 7.18 | -0.13 | 0.02 | -0.10 | 0.14 | 0.10 | 0.19 | 0.16 |
| Std error | 0.120 | 0.040 | 0.002 | 0.132 | 0.058 | 0.053 | 0.070 | 0.059 |
| | Earners | HHsize | Occ. 0 | Occ. 1 | Occ. 4 | Sch. 10 | Sch. 12 | Sch. 16 |
| All HHs | 0.25 | -0.14 | 0.44 | 0.23 | 0.15 | -0.36 | -0.14 | 0.16 |
| Std error | 0.017 | 0.006 | 0.038 | 0.038 | 0.033 | 0.046 | 0.030 | 0.029 |
| Respond. | 0.27 | -0.14 | 0.45 | 0.26 | 0.15 | -0.36 | -0.14 | 0.19 |
| Std error | 0.029 | 0.012 | 0.064 | 0.067 | 0.050 | 0.074 | 0.048 | 0.050 |

The numbers attached to the variable “Dist.” define different districts of residence. The numbers attached to the variable “Occ.” define different classes of occupations. The numbers attached to the variable “Sch.” define the number of years at school.

Table 2. Coefficients of logistic model for response probabilities and standard errors when fitting the model to all the sampled HHs (Respondents and Nonrespondents), and when fitting the model to only the responding HHs

| | Const. | Log(y) | Gender | Dist. 43 | Dist. 44 | Dist. 53 | HH size |
|-----------|--------|------------|--------|----------|----------|----------|---------|
| All HHs | 1.00 | -0.21 | -0.21 | 0.86 | -0.58 | -0.77 | 0.10 |
| Std error | 0.828 | 0.088 | 0.109 | 0.227 | 0.190 | 0.275 | 0.033 |
| Respond. | 1.35 | -0.26 | -0.20 | 0.90 | -0.59 | -0.79 | 0.12 |
| Std error | 1.317 | 0.141 | 0.141 | 0.353 | 0.225 | 0.318 | 0.047 |

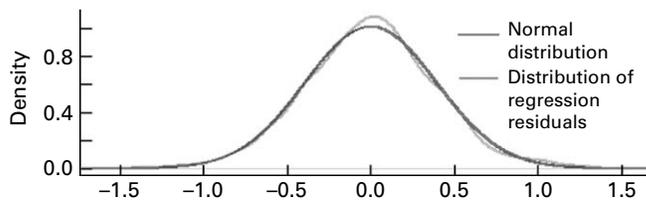


Fig. 1. Histogram of estimated regression residuals and normal density with mean zero and same variance ($\hat{\sigma}_e^2 = 0.155$)

almost three times as large and the two models are fitted separately. When fitting the sample model (Equation 35) to the entire sample, we obtain $R^2 = 0.612$ with residual variance $\hat{\sigma}_e^2 = 0.155$. The estimate of σ_e^2 when fitting the model to only the responding units is $\hat{\sigma}_e^2 = 0.154$. The values of the regression coefficients are sensible. For example, the coefficients of the education variables increase as the level of education increases. The number of earners in the household has a strong positive effect on the income, while the size of the household has a strong negative effect. The coefficient of Gender (being a female) is negative.

Figure 1 compares the histogram of the estimated sample model residuals when fitting the model to all the sample data, with a normal *pdf* with mean zero and same variance, $\hat{\sigma}_e^2 = 0.155$. The distribution of the estimated residuals is seen to be close to the normal distribution, although with somewhat shorter tails, which can be explained by the fact that the estimated residuals are not independent. The normality assumption is validated in Section 7.5 based on fitting the model to only the responding units.

7.3. Imputation of Missing Outcomes

Next we show the performance of the proposed approach in imputing the missing incomes. The imputations were carried out under two scenarios. In Scenario 1 we use the known covariates for the nonrespondents and impute the incomes by drawing at random from the estimated *pdf* $\hat{f}_{R^c}(y_i|z_i) = f(y_i|z_i, i \in S, R_i = 0; \hat{\beta}, \hat{\sigma}_e^2, \hat{\gamma})$ (Eq. 20). We imputed five values for each unit and averaged the five imputations. In Scenario 2 the covariates for the nonresponding units are also imputed using Equation (26), and then the incomes are imputed similarly to Scenario 1. Figure 2 compares the true empirical cumulative distribution of the incomes of the nonresponding units with the means of the empirical distributions of the imputed values over the five imputation sets. Also shown is the cumulative distribution of the imputed incomes when ignoring the nonresponse process,

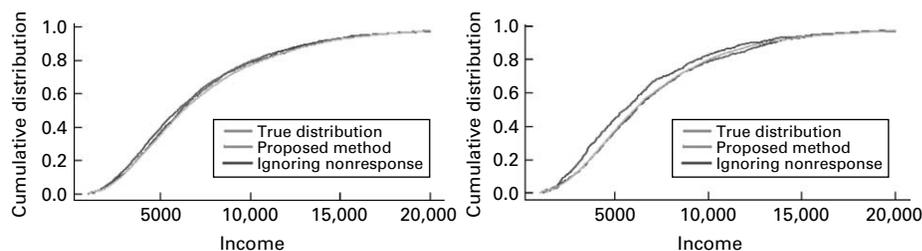


Fig. 2. True empirical cumulative distribution and means of empirical cumulative distributions of the imputed incomes over five imputation sets when the covariates for the nonresponding units are known (left panel) and when they are unknown (right panel)

Table 3. Distribution of HH size in subsamples of responding and nonresponding HHs

| HH size | 1 | 2 | 3 | 4 | 5 | 6 + |
|-------------|-------|-------|-------|-------|-------|-------|
| Respond. | 6.18 | 13.63 | 19.33 | 26.94 | 20.60 | 13.31 |
| Nonrespond. | 12.39 | 19.00 | 17.34 | 24.40 | 17.34 | 9.54 |

that is, when imputing the missing covariates by drawing at random from their empirical distribution for the responding HHs and imputing the missing incomes given the covariates by drawing at random from the estimated sample distribution. This distribution is again a mean over five imputation sets.

Figure 2 shows that application of the proposed approach yields imputations with distribution that is close to the true distribution. On the other hand, ignoring the nonresponse yields biased imputations, particularly when the covariates for the nonresponding units are likewise unknown.

It is important to emphasize that even if the distribution of the income given the covariates was the same for the responding and nonresponding HHs, ignoring the nonresponse in the case of unknown covariates for the nonresponding units would still produce biased estimates for the income distribution. This is because the nonresponse process cannot be ignored for some of the covariates. For example, Table 3 shows the percentage of HHs by size for the responding and nonresponding units. The HH size is an important covariate in Models (35) and (36) (Tables 1 and 2).

7.4. Estimation of Mean Sample Income and Variance of Estimators

In Section 4 we considered three estimators of the population mean of the outcome variable and in Section 5 we considered alternative ways of estimating their variance. Tables 4 and 5 summarize the results obtained when estimating the true sample mean of the incomes. Table 4 presents the estimated standard errors (S.E.) when conditioning on the observed covariates for the respondents (and hence also on the number of respondents). Table 5 presents the unconditional S.E. estimators. For both cases we used bootstrap samples as described in Section 5. Also shown in the two tables is the mean and S.E. over all bootstrap samples of the H-T estimator that uses the “true” probabilities of responding, $\pi(y_i, v_i; \hat{\gamma})$, that is, when the probabilities of responding are not reestimated from the

Table 4. Estimation of sample mean of income (True $\bar{Y} = 7215.06$). Conditional S.E. 500 bootstrap samples

| Estimator | Estimate | | Standard error |
|---------------------------|--------------------------------|-----------------------------|----------------|
| | Original sample of respondents | Mean over bootstrap samples | |
| $\hat{\bar{Y}}_{(1,P-K)}$ | – | 7,303.65 | 174.16 |
| $\hat{\bar{Y}}_{(1)}$ | 7,332.30 | 7,299.17 | 147.38 |
| $\hat{\bar{Y}}_{(2)}$ | 7,311.06 | 7,297.09 | 140.81 |
| $\hat{\bar{Y}}_{(3)}$ | 7,272.26 | 7,265.53 | 146.58 |

Table 5. Estimation of sample mean of income (True $\bar{Y} = 7215.06$). Unconditional S.E. 500 bootstrap samples

| Estimator | Estimate | | Standard error |
|---------------------------|--------------------------------|-----------------------------|----------------|
| | Original sample of respondents | Mean over bootstrap samples | |
| $\hat{\bar{Y}}_{(1,P-K)}$ | – | 7,246.26 | 347.39 |
| $\hat{\bar{Y}}_{(1)}$ | 7,332.30 | 7,248.88 | 179.83 |
| $\hat{\bar{Y}}_{(2)}$ | 7,311.06 | 7,308.37 | 148.00 |
| $\hat{\bar{Y}}_{(3)}$ | 7,272.26 | 7,304.99 | 152.43 |

bootstrap samples. This estimator, denoted by $\hat{\bar{Y}}_{(1,P-K)}$, does not take into account the known totals of the covariates via the calibration constraints.

Tables 4 and 5 show that all the estimators of the mean population income overestimate the true mean, but with the largest bias in the two tables being 1.6%. In comparison, the mean of the incomes computed from only the responding units is 6,822.42, an underestimation of 5.4%. As anticipated, the standard errors of the estimators are smaller when conditioning on the observed covariates (Table 4) than in the case where the standard errors are taken over all possible samples of respondents (Table 5). Also, the standard errors are somewhat smaller when the covariates for the nonresponding units are known (the estimator $\hat{\bar{Y}}_{(2)}$) than in the case that they have to be imputed (the estimator $\hat{\bar{Y}}_{(3)}$). Finally, the estimator $\hat{\bar{Y}}_{(1,P-K)}$, which does not use the calibration constraints, has a much larger variance than the estimator $\hat{\bar{Y}}_{(1)}$ illustrating the advantage of modifying the weights by use of calibration constraints.

For estimating the unconditional standard error of the H-T estimator $\hat{\bar{Y}}_{(1)}$ we also computed for each of the 500 bootstrap samples the estimator (29), using the distribution over all possible samples of respondents and all possible outcomes. The mean of the S.E. estimates turned out to be 184.78, which is very close to the empirical standard error of 179.83 over all the bootstrap samples. The standard error estimator based on the original sample is 185.24.

7.5. Testing the Model Assumptions

In this section we study the performance of the test statistics (30) – (34) by generating new original samples from three different distributions of the sample model residuals and

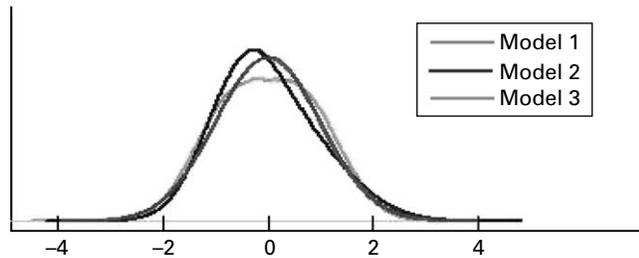


Fig. 3. Sample models used for studying the performance of the test statistics

fitting in each case the respondents' model, which assumes that the residuals have a $N(0, \sigma_\varepsilon^2)$ distribution:

- I. The true residual distribution is $N(0, \hat{\sigma}_\varepsilon^2)$ (Model 1); in this case the fitted model is correct. Here and below $\hat{\sigma}_\varepsilon^2 = 0.155$, as estimated for the actual sample.
- II. The true residual distribution is a mixture of $N(0.5\hat{\sigma}_\varepsilon^2, \hat{\sigma}_\varepsilon^2)$ and $N(-0.5\hat{\sigma}_\varepsilon^2, 0.5\hat{\sigma}_\varepsilon^2)$ with equal probabilities (Model 2); in this case the fitted model is incorrect.
- III. The true residual distribution is a mixture of $N(0.7\hat{\sigma}_\varepsilon^2, 0.51\hat{\sigma}_\varepsilon^2)$ and $N(-0.7\hat{\sigma}_\varepsilon^2, 0.51\hat{\sigma}_\varepsilon^2)$ with equal probabilities (Model 3); in this case the fitted model is incorrect.

In all the three cases we sampled the respondents using the logistic model, which was assumed also under the misspecified distributions. Figure 3 shows the densities of the residuals under the three sample models.

In order to study the performance of the test statistics we performed the following experiment for each of the three models defined above:

1. Generate 250 samples of outcomes from the respondents' distribution under the model with fixed (original) covariates z_i and parameters $(\hat{\theta}, \hat{\gamma})$ estimated from the original true sample of the respondents (Tables 1 and 2).
2. For each new sample:
 - 2i. Estimate (θ, γ) assuming that the true sample distribution of the residuals is normal,
 - 2ii. Compute the test statistics (30) – (34),
 - 2iii. Generate 250 new samples from the respondents' distribution assuming that the sample distribution of the residuals is normal, using the estimated parameters from (2i). Then,
3. For each new sample generated in 2iii,
 - 3i. Reestimate (θ, γ) assuming that the true sample distribution is normal and compute the test statistics (30) – (34).
 - 3ii. Compute the distribution of each test statistic based on the 250 values in 3i.

Notice that Models I – III are only used for generating the new original samples. The test statistics in 2ii use these samples while the distribution of the test statistics in 3ii is computed (empirically) based on the newly generated samples in 2iii.

Table 6 compares the empirical distribution of the five test statistics under Model 1 as obtained in Step 2ii, with the nominal values computed in Step 3ii. Denoting the ordered

Table 6. Empirical and theoretical distribution of test statistics under Model 1

| Test | Nominal levels | | | | | | | | | | | |
|-----------|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| | 0.05 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 1.00 |
| KS | 0.040 | 0.046 | 0.088 | 0.079 | 0.109 | 0.097 | 0.112 | 0.096 | 0.146 | 0.097 | 0.039 | 0.053 |
| AD | 0.048 | 0.046 | 0.068 | 0.095 | 0.093 | 0.113 | 0.109 | 0.144 | 0.074 | 0.109 | 0.043 | 0.058 |
| CM | 0.056 | 0.031 | 0.084 | 0.110 | 0.072 | 0.080 | 0.125 | 0.133 | 0.097 | 0.101 | 0.059 | 0.051 |
| $C_{(3)}$ | 0.043 | 0.035 | 0.088 | 0.120 | 0.109 | 0.120 | 0.088 | 0.107 | 0.079 | 0.102 | 0.055 | 0.053 |
| $C_{(4)}$ | 0.042 | 0.043 | 0.060 | 0.126 | 0.098 | 0.135 | 0.143 | 0.108 | 0.093 | 0.092 | 0.027 | 0.053 |

Table 7. Proportion of samples that each test statistic rejects the misspecified model for different nominal significance levels

| Test | Model 2 | | | | Model 3 | | | |
|-----------|--------------------|-------|-------|-------|--------------------|-------|-------|-------|
| | Significance level | | | | Significance level | | | |
| | 0.01 | 0.025 | 0.05 | 0.10 | 0.01 | 0.025 | 0.05 | 0.10 |
| KS | 0.832 | 0.892 | 0.936 | 0.960 | 0.245 | 0.549 | 0.637 | 0.775 |
| AD | 0.936 | 0.964 | 0.984 | 0.988 | 0.588 | 0.725 | 0.784 | 0.853 |
| CM | 0.924 | 0.948 | 0.980 | 0.988 | 0.490 | 0.696 | 0.765 | 0.843 |
| $C_{(3)}$ | 0.876 | 0.932 | 0.956 | 0.984 | 0.000 | 0.000 | 0.020 | 0.088 |
| $C_{(4)}$ | 0.112 | 0.188 | 0.264 | 0.356 | 0.480 | 0.647 | 0.716 | 0.823 |

values of any one of the test statistics obtained in Step 3ii by $u_{(1)} < \dots < u_{(250)}$, the critical value for nominal level α_j was defined as $u_{(250\alpha_j)}$ when $250\alpha_j$ is an integer, and $u_{[250\alpha_j]+1}$ otherwise, where $[\cdot]$ defines the integer number. The value of any given statistic in a cell corresponding to nominal level α_j is the percentage of samples that the statistic computed in Step 2ii was between the critical values for nominal levels α_{j-1} and α_j computed in Step 3ii ($\alpha_0 = 0$).

In general, the empirical distribution of all the statistics is sufficiently close to the nominal values, thus validating the parametric bootstrap procedure described in Section 6 for calculating the critical values under an assumed model. The goodness of fit of the empirical distributions to the nominal values was tested by the Pearson Chi-square statistic with 11 degrees of freedom, yielding p -values of 0.67 for KS, 0.45 for AD, 0.47 for CM, 0.89 for $C_{(3)}$ and 0.13 for $C_{(4)}$.

Table 7 shows the proportion of samples that each test statistic rejects the misspecified distribution for the responding units, which assumes that the sample distribution of the model residuals is normal when in fact the true sample distribution is as defined under Model 2 or Model 3. For the test statistics defined by (30) – (32) we used one-sided tests. For the test statistics defined by (33) – (34) we used two-sided tests. The proportions in Table 7 estimate the powers of the various tests in rejecting the misspecified model.

When the true sample distribution is skewed as under Model 2, the three classical test statistics and the statistic $C_{(3)}$ that is designed for testing the skewness of the distribution have very good power properties, with powers higher than 0.9 for significance levels equal to or higher than 0.025. As could be anticipated, the test $C_{(4)}$, which is designed to test the fourth moment, has very low power in this case. When the true sample distribution is symmetric but flatter than the normal distribution, as under Model 3, the powers of all the test statistics except $C_{(4)}$ reduce. Nonetheless, all the test statistics except $C_{(3)}$, and in particular AD and CM still have acceptable powers in this case for significance levels equal to or higher than 0.05.

The power of test statistics depends on the distance between the true model and the misspecified model and as shown by Figure 3, the distances in our case are not really large. Although more work is needed for developing new test procedures, the results in Tables 6 and 7 suggest that the goodness of fit of models fitted for the responding units can be tested adequately.

Table 8. *P-values when testing the model fitted to the original sample (Equations 35 – 36)*

| Test | KS | AD | CM | $C_{(3)}$ | $C_{(4)}$ |
|-----------------|-------|-------|-------|-----------|-----------|
| <i>p</i> -value | 0.262 | 0.098 | 0.122 | 0.256 | 0.108 |

Finally, we applied the test procedure described above (Steps 2ii and onwards) in order to validate the normal/logistic model fitted to the original household expenditure data (Equations (35)–(36)). For this, we generated 500 samples from the respondents’ distribution with the parameter estimates in Tables 1 and 2 obtained when fitting the model to only the responding units. Table 8 exhibits the *p*-values obtained for the five test statistics.

With the usual type II error in mind and noticing the powers exhibited in Table 7 under the misspecified models, the *p*-values in Table 8 support the normal/logistic model fitted to the data, as already suggested by the other empirical results shown in previous sections.

8. Summary

NMAR nonresponse is hard to deal with and inevitably requires the use of models. In this article we develop a comprehensive modelling approach, covering all the necessary steps from parameter estimation and model testing to imputation of the missing covariates and outcomes, and estimation of population means and the variance of the estimators. We illustrate all the steps by analyzing a real data set, illustrating that the approach can be applied very successfully.

As with any new method, there is even here need for extensions. In this article we assumed that the sample selection is noninformative and that the outcomes are independent. Accounting for informative sampling under the present approach can be implemented by modifying the sample distribution along the lines of Pfeffermann and Sverchkov (2003). Accounting for correlated observations as under two-stage cluster sampling will be more involved and require several modifications to the approach, such as in variance estimation and model testing. As a final possible extension we mention the use of the proposed approach in a Bayesian set-up, thus benefitting from the extra flexibility of Bayesian inference. An interesting problem arising in this respect is how to incorporate the calibration equations (18) in the Bayesian paradigm. We have started working in this direction.

APPENDIX A: Solution of Equations (15a) – (15b)

In order to solve Equations (15a) – (15b) for a given vector coefficient γ we use the Newton-Raphson algorithm. The second derivatives are as follows:

$$\frac{\partial^2 l_j}{\partial \beta_k^2} = -a(\phi) \sum_{i=1}^r \text{Var}_R(Y_i | z_i; \beta, \phi, \gamma) x_{ji} x_{ki}, \quad j, k = 0, \dots, p$$

$$\frac{\partial l_j}{\partial \phi} = -\frac{\partial a(\phi)}{\partial \phi} \left\{ \sum_{i=1}^r \left(\text{Var}_R(Y_i|z_i; \beta, \phi, \gamma) \sum_{s=0}^p \beta_s x_{si} + \text{COV}_R(Y_i, d(Y_i)|z_i; \beta, \phi, \gamma) \right) x_{ki} \right\}$$

$$j = 0, \dots, p$$

$$\frac{\partial l_{p+1}}{\partial \beta_k} = -a(\phi) \sum_{i=1}^r \text{COV}_R(Y_i, d(Y_i)|z_i; \beta, \phi, \gamma) x_{ki}, k = 0, \dots, p$$

$$\frac{\partial l_{p+1}}{\partial \phi} = -\frac{\partial a(\phi)}{\partial \phi} \left\{ \sum_{i=1}^r \left(\text{Var}_R(d(Y_i)|z_i; \beta, \phi, \gamma) + \text{COV}_R(Y_i, d(Y_i)|z_i; \beta, \phi, \gamma) \sum_{s=0}^p \beta_s x_{si} \right) \right\}$$

where Var_R and Cov_R are the variance and covariance with respect to the distribution holding for the responding units. Denote

$$X' = \begin{bmatrix} 1, \dots, 1, & 0, \dots, 0 \\ x_1, \dots, x_r, & 0, \dots, 0 \\ 0, \dots, 0 & 1, \dots, 1 \end{bmatrix}, \quad Y = [y_1, \dots, y_r, d(y_1), \dots, d(y_r)]'$$

$$\mu' = \{E_R(Y_1|z_1), \dots, E_R(Y_r|z_r), E_R[d(Y_1)|z_1], \dots, E_R[d(Y_r)|z_r]\}$$

$$V_1 = \text{Diag}\{\text{Var}_R(Y_i|z_i)\}, V_2 = \text{Diag}\{\text{Var}_R(d(Y_i)|z_i)\}, i = 1, \dots, r$$

$$C = \text{Diag}\{\text{COV}_R((d(Y_i), Y_i)|z_i)\}, S = \text{Diag}\left[\sum_{k=0}^p \beta_k x_{ki}\right], i = 1, \dots, r$$

Application of the Newton-Raphson method yields

$$\begin{pmatrix} \beta \\ \phi \end{pmatrix}^{(m+1)} = \begin{pmatrix} \beta \\ \phi \end{pmatrix}^{(m)} - [A^{-1}(X^T W X)^{-1} X^T (Y - \mu)] \quad (\text{A1})$$

where A is a diagonal matrix of dimension $(p+2) \times (p+2)$, with all the elements on the main diagonal being $-a(\phi)$, except for the last element which is $-\partial a(\phi)/\partial \phi$ and

$$W = \begin{bmatrix} V_1 & C + S V_1 \\ C & V_2 + S C \end{bmatrix}$$

APPENDIX B: Proof of Theorem 1

We need to solve the equations $U(\xi) = 0$, where $\xi' = (\theta', \gamma') = (\theta_0, \dots, \theta_{p+1}, \gamma_0, \dots, \gamma_{q+1})$ and $U = (l', h') = (l_0, \dots, l_{p+1}, h_0, \dots, h_{q+1})'$. The functions $l(\xi) = [l_0(\xi), \dots, l_{p+1}(\xi)]'$ are defined by (15a)–15(b) with $\theta' = (\beta', \phi)$. The functions $h(\xi) = [h_0(\xi), \dots, h_{q+1}(\xi)]'$ are defined right before Theorem 1 in Section 3.4. We now show that the solution $\hat{\xi}$ of the algorithm of Section 3.4 converges in probability to the solution of the equations $U(\xi) = 0$, which we denote by $\xi_0 = (\theta'_0, \gamma'_0)'$.

Consider the familiar Newton-Raphson algorithm. Application of this algorithm to the present problem requires computing iteratively until convergence the estimators

$$\xi^{(m+1)} = \xi^{(m)} - A_m^{-1} U(\xi^{(m)}) \quad (\text{B1})$$

where $\xi^{(m)}$ is the solution on the m th iteration. The matrix A_m is defined as,

$$A_m = \begin{pmatrix} \frac{\partial l(\theta^{(m)}, \gamma^{(m)})}{\partial \theta^{(m)}} & \frac{\partial l(\theta^{(m)}, \gamma^{(m)})}{\partial \gamma^{(m)}} \\ \frac{\partial h(\theta^{(m)}, \gamma^{(m)})}{\partial \theta^{(m)}} & \frac{\partial h(\theta^{(m)}, \gamma^{(m)})}{\partial \gamma^{(m)}} \end{pmatrix}, \text{ where } \frac{\partial l(\theta^{(m)}, \gamma^{(m)})}{\partial \theta^{(m)}}$$

is the matrix of partial derivatives of $l(\theta, \gamma)$ with respect to θ evaluated at $(\theta^{(m)}, \gamma^{(m)})$, and similarly for the other block matrices. The estimation algorithm in Section 3.4 splits instead the system $U(\xi) = 0$ into the two systems, $l(\xi) = 0$ and $h(\xi) = 0$, and solves them iteratively until convergence as follows:

Apply one Newton-Raphson iteration to the equations $l(\xi) = 0$ with respect to θ for given γ , and one Newton-Raphson iteration to the equations $h(\xi) = 0$ with respect to γ for given θ , where the given values of θ and γ are the solutions from the previous iteration. The updating equations in this case can be written as,

$$\xi^{(m+1)} = \xi^{(m)} - B_m^{-1} U(\xi^{(m)}) \quad (\text{B2})$$

$$\text{where } B_m = \begin{pmatrix} \frac{\partial l(\theta^{(m)}, \gamma^{(m)})}{\partial \theta^{(m)}} & 0 \\ 0 & \frac{\partial h(\theta^{(m)}, \gamma^{(m)})}{\partial \gamma^{(m)}} \end{pmatrix}$$

Splitting the system of equations into the two subsystems is advantageous for large dimensional systems, since it saves the computation of $2(p+2) \times (q+2)$ (possibly complicated) partial derivatives. Define matrices M_{01} and M_{02} as follows:

$$M_{01} = \left[\frac{\partial l(\theta_0, \gamma_0)}{\partial \theta_0} \right]^{-1} \frac{\partial l(\theta_0, \gamma_0)}{\partial \gamma_0} \left[\frac{\partial h(\theta_0, \gamma_0)}{\partial \gamma_0} \right]^{-1} \frac{\partial h(\theta_0, \gamma_0)}{\partial \theta_0}$$

$$M_{02} = \left[\frac{\partial h(\theta_0, \gamma_0)}{\partial \gamma_0} \right]^{-1} \frac{\partial h(\theta_0, \gamma_0)}{\partial \theta_0} \left[\frac{\partial l(\theta_0, \gamma_0)}{\partial \theta_0} \right]^{-1} \frac{\partial l(\theta_0, \gamma_0)}{\partial \gamma_0}$$

Suppose now that the conditions of Theorem 1 hold and that $\lim_{N \rightarrow \infty, n \rightarrow \infty} P \|M_{01}\| = \lambda_{01} < 1$, $\lim_{N \rightarrow \infty, n \rightarrow \infty} P \|M_{02}\| = \lambda_{02} < 1$, where $\|\cdot\|$ defines the Euclidian norm. We later check the fulfillment of the conditions for the model used for the empirical study in Section 7.

Proof of Theorem 1 It is known that the Newton-Raphson algorithm has a quadratic rate of convergence, implying $\|\varepsilon^{(m+1)}\| < c \|\varepsilon^{(m)}\|^2$, where $\varepsilon^{(m)} = (\xi^{(m)} - \xi_0)$ and c is a constant. Thus, by (B1),

$$\|\varepsilon^{(m)} - A_m^{-1} U(\xi^{(m)})\| < c \|\varepsilon^{(m)}\|^2 \quad (\text{B3})$$

Next, rewrite the equations B(2) as $\xi^{(m+1)} = \xi^{(m)} - B_m^{-1}A_m A_m^{-1}U(\xi^{(m)})$. The rate of convergence of the proposed algorithm can be derived therefore as,

$$\begin{aligned} \varepsilon^{(m+1)} &= \varepsilon^{(m)} - B_m^{-1}A_m A_m^{-1}U(\xi^{(m)}) \\ &= \varepsilon^{(m)} + B_m^{-1}A_m((\varepsilon^{(m)} - A_m^{-1}U(\xi^{(m)})) - \varepsilon^{(m)}) \\ &= (I - B_m^{-1}A_m)\varepsilon^{(m)} + B_m^{-1}A_m(\varepsilon^{(m)} - A_m^{-1}U(\xi^{(m)})), \text{ or,} \end{aligned} \tag{B4}$$

$$\|\varepsilon^{(m+1)}\| \leq \|I - B_m^{-1}A_m\| \cdot \|\varepsilon^{(m)}\| + \|B_m^{-1}A_m\| \cdot \|\varepsilon^{(m)} - A_m^{-1}U(\xi^{(m)})\|$$

By (B3), $\|\varepsilon^{(m+1)}\| \leq \|I - B_m^{-1}A_m\| \cdot \|\varepsilon^{(m)}\| + c\|B_m^{-1}A_m\| \cdot \|\varepsilon^{(m)}\|^2$ and hence for $\varepsilon^{(m)}$ sufficiently small,

$$\|\varepsilon^{(m+1)}\| \leq \|H_m\| \cdot \|\varepsilon^{(m)}\| \tag{B5}$$

where $H_m = H(\theta^{(m)}, \gamma^{(m)}) = I - B_m^{-1}A_m$. It follows that,

$$\|\varepsilon^{(m+1)}\| \leq \|H_{m-1}H_m\| \cdot \|\varepsilon^{(m-1)}\| \tag{B6}$$

$$\text{Now, } H_m = \begin{bmatrix} 0_{(p+2) \times (p+2)} & -\left[\frac{\partial l(\theta^{(m)}, \gamma^{(m)})}{\partial \theta^{(m)}}\right]^{-1} \frac{\partial l(\theta^{(m)}, \gamma^{(m)})}{\partial \gamma^{(m)}} \\ -\left[\frac{\partial h(\theta^{(m)}, \gamma^{(m)})}{\partial \gamma^{(m)}}\right]^{-1} \frac{\partial h(\theta^{(m)}, \gamma^{(m)})}{\partial \theta^{(m)}} & 0_{(q+2) \times (q+2)} \end{bmatrix}$$

and let $H_0 = H(\theta, \gamma)|_{(\theta_0, \gamma_0)}$. By Taylor expansion of H_m around (θ_0, γ_0) , $H_m \approx H_0 + D_0E^{(m)}$, where the elements of D_0 and $E^{(m)}$ are defined as follows: Let h_{ij} be the derivative $\partial H_{ij} / \partial \xi_k$, where H_{ij} denotes the (i, j) th element of the matrix H_0 . The matrix D_0 is obtained from H_0 by replacing each element H_{ij} by the row vector $(h_{ij0}, \dots, h_{ijp}, h_{ij,p+1}, \dots, h_{ij,p+q+3})$. The matrix $E^{(m)}$ is $E^{(m)} = I_{p+q+4} \otimes \varepsilon^{(m)}$ where \otimes defines the Kronecker product. Note that $\dim(D_0) = (p + q + 4) \times (p + q + 4)^2$, and $\dim(E^{(m)}) = (p + q + 4)^2 \times (p + q + 4)$. It follows that,

$$\begin{aligned} \|\varepsilon^{(m+1)}\| &\leq \|H_{m-1}H_m\| \cdot \|\varepsilon^{(m-1)}\| \approx \\ \|(H_0 + D_0E^{(m-1)})(H_0 + D_0E^{(m)})\| \cdot \|\varepsilon^{(m-1)}\| &\approx \|H_0^2\| \cdot \|\varepsilon^{(m-1)}\|, \text{ and hence by (B6),} \end{aligned}$$

$$\|\varepsilon^{(m+1)}\| \leq \|H_0^2\|^k \cdot \|\varepsilon^{(m-2k+1)}\| \tag{B7}$$

$$\text{where } H_0^2 = \begin{bmatrix} M_{01} & 0 \\ 0 & M_{02} \end{bmatrix}$$

Since we assume, $\lim_{N \rightarrow \infty, n \rightarrow \infty} P \|M_{01}\| = \lambda_{01} < 1$, $\lim_{N \rightarrow \infty, n \rightarrow \infty} P \|M_{02}\| = \lambda_{02} < 1$, we obtain $\lim_{N \rightarrow \infty, n \rightarrow \infty} P \|H_0^2\|^k = 0$ as $k \rightarrow \infty$, and

$$\|\varepsilon^{(m+1)}\| = \|\xi^{(m+1)} - \xi_0\| \xrightarrow{L} 0 \text{ for } m - 2k \rightarrow \infty, k \rightarrow \infty \tag{B8}$$

showing that the solution $\xi^{(m)}$ of the proposed algorithm converges in probability to ξ_0 . QED

It remains to show that the conditions of the theorem are satisfied by Equations (15a) – (15b) and (18a) – (18b) as obtained for the model defined by (35) – (36). It is easy to show that in this case the functions $l(\theta, \gamma)$ and $h(\theta, \gamma)$ satisfy the conditions I) and III), provided that $0 < \pi(y_i, v_i; \gamma) < 1$ with bounded first derivatives (Condition II). For example, in (36) $\pi(y_i, v_i; \gamma)$ is the logistic probability and denoting

$$v_{0,i} = 1, v_{q+1,i} = y_i, \frac{\partial \pi(y_i, v_i; \gamma)}{\partial \gamma_l} = \frac{v_{li} e^{-(v'_i \gamma + \gamma_0 v_{q+1,i})}}{(1 + e^{-(v'_i \gamma + \gamma_0 v_{q+1,i})})^2} \text{ for } l = 0, \dots, (q + 1)$$

In order to show that the norms of M_{01} and M_{02} converge in probability to limits smaller than 1, we assume (r1) $\sum_{i=1}^r w_i = O(N^\delta)$ for $0.5 < \delta \leq 1$, (r2) $\sum_{j=1}^N w_j = O(N)$ and

$$(r3) \text{ Var} \left[\sum_{i=1}^r w_i \frac{x_{ki}}{\pi(y_i, v_i; \gamma)} \right] = O(N)$$

These are standard requirements in sample surveys. As a simple example suppose that the sample is drawn by simple random sampling without replacement and the response probabilities are $\pi(y_i, v_i; \gamma) = (r/n)$. In this case, $w_i = (N/n)$ and $\sum_{j=1}^N w_j = N^2/n = O(N)$ since we assume that (N/n) is bounded. Clearly, $\sum_{i=1}^r w_i = r(N/n) = O(N^\delta)$ for $0.5 < \delta \leq 1$ and the condition (r3) is also satisfied as long as $(N/r) = (N/n)(n/r)$ is bounded.

Suppose for convenience that $\dim(x_i) = 2$ and $\dim(v_i) = 1$. Then, $\theta = (\beta_0, \beta_1, \beta_2, \phi)'$ and $\gamma = (\gamma_0, \gamma_1, \gamma_2)'$. The matrix

$$\left. \frac{\partial h(\theta, \gamma)}{\partial \theta} \right|_{(\theta_0, \gamma_0)}$$

for the functions $h(\theta, \gamma)$ in (18a)–(18b) is then a 3×4 matrix with all of its elements equal to zero except for the (3,3)th element, which equals

$$\sum_{i=1}^r w_i \frac{\tilde{x}_i}{\pi(y_i, v_i; \gamma)} - \tilde{X}^{pop}$$

Next consider the derivatives of the functions in $h(\theta, \gamma)$ with respect to γ . Denoting as above $v_{0,i} = 1, v_{q+1,i} = y_i$, we have that

$$\frac{\partial}{\partial \gamma_l} \sum_{i=1}^r w_i \frac{1}{\pi(y_i, v_i; \gamma)} = - \sum_{i=1}^r w_i \frac{1}{\pi(y_i, v_i; \gamma)^2} \frac{\partial \pi(y_i, v_i; \gamma)}{\partial \gamma_l} \text{ } l = 0, \dots, (q + 1)$$

Note that under the logistic model

$\frac{1}{\pi(y_i, v_i; \gamma)^2} \frac{\partial \pi(y_i, v_i; \gamma)}{\partial \gamma_l} = O(1)$ and $\sum_{i=1}^r w_i = O(N^\delta)$ by (r1) and therefore the whole expression is $O(N^\delta)$. Some further algebra shows that the only nonzero entry of the matrix

$$\left[\frac{\partial h(\theta, \gamma)}{\partial \gamma} \right]^{-1} \frac{\partial h(\theta, \gamma)}{\partial \theta}$$

is a constant times

$$N^{-\delta} \left[\sum_{i=1}^r w_i \frac{\tilde{x}_i}{\pi(y_i, v_i; \gamma)} - \tilde{X}^{pop} \right]$$

and by Chebyshev inequality and the condition (r3),

$$P \left[N^{-\delta} \left(\left| \sum_{i=1}^r w_i \frac{\tilde{x}_i}{\pi(y_i, v_i; \gamma)} - \tilde{X}^{pop} \right| > \varepsilon \right) \right] \xrightarrow[N \rightarrow \infty]{P} 0, \text{ if } \delta > 0.5$$

It follows that all the elements of

$$\left[\frac{\partial h(\theta, \gamma)}{\partial \gamma} \right]^{-1} \frac{\partial h(\theta, \gamma)}{\partial \theta} \xrightarrow[N \rightarrow \infty]{P} 0$$

and hence the norms of M_{01} and M_{02} converge in probability to limits smaller than 1, as assumed for the proof.

APPENDIX C: Proof of Theorem 2

The consistency of the estimator follows from a result by Huber (1967), which states that under the conditions of the theorem, as $n \rightarrow \infty$ the estimator ξ_0 solving the equations $U(\xi) = 0$ is consistent for $\tilde{\xi}$, the solution of the equations $E[U(\xi)] = 0$. By Theorem 1, the estimator $\hat{\xi}$ converges in probability to ξ_0 , establishing its consistency under the same conditions.

The asymptotic normality of the estimator $\hat{\xi}$ follows from a result by Newey and McFadden (1994), which states that under the same conditions,

$$\sqrt{n}(\hat{\xi} - \tilde{\xi}) \xrightarrow{D} N[0, V(\tilde{\xi})] \quad (C1)$$

with the variance matrix $V(\tilde{\xi})$ defined as $V(\tilde{\xi}) = A(\tilde{\xi})^{-1} B(\tilde{\xi}) [A(\tilde{\xi})^{-1}]'$, where by abbreviating $\varphi_i = \varphi(R_i, y_i, z_i, \tilde{\xi})$, $A_n(\tilde{\xi}) = 1/n \sum_{i=1}^n E(-\nabla \varphi_i)$, $B_n(\tilde{\xi}) = 1/n \sum_{i=1}^n E(\varphi_i \varphi_i')$; $A(\tilde{\xi}) = \lim_{n \rightarrow \infty} A_n(\tilde{\xi})$, $B(\tilde{\xi}) = \lim_{n \rightarrow \infty} B_n(\tilde{\xi})$ and $\nabla \varphi_i$ is the matrix of first derivatives of φ_i with respect to ξ .

Equations (15a) – (15b) and (18a) – (18b) satisfy the conditions of the theorem, thus establishing the consistency and asymptotic normality of the estimator $\hat{\xi}$. *QED.*

9. References

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