Journal of Official Statistics, Vol. 12, No. 2, 1996, pp. 199–222

# A Comparison of Ten Methods for Multilateral International Price and Volume Comparison

*Bert M. Balk*[1]

In this article we compare the performance of a number of methods for calculating (transitive) purchasing power parities and volume indices with respect to a set of tests proposed by Diewert (1986). The methods considered are: the Van IJzeren (balanced) method, the YKS and Q-YKS methods, the weighted EKS-Fisher method, the weighted Own Share system, the methods of Geary-Khamis, Iklé, Gerardi, the KS-S method, and the Rao method. It appears that, with respect to the number of tests failed, the Van IJzeren method and the Geary-Khamis method show superior performance to the other methods. The article closes with a discussion of additivity.

*Key words:* Index number theory; purchasing power parities; international comparison; additivity.

## 1. Introduction

For a variety of purposes it is important to make international comparisons of (components of) Gross Domestic Product or other important figures such as per capita income, or the value of industrial output. It is widely recognized that converting the value of such economic aggregates into a *numéraire* currency by means of exchange rates yields very unsatisfactory results. Index numbers based on observed prices and quantities, corresponding to the internal structure of the aggregates under study, should be used. In the course of time, quite a number of methods for calculating such index numbers have been developed. When one wishes to compare only two countries at a time (a so-called bilateral comparison), one can simply borrow methods familiar from the field of intertemporal comparisons. However, multilateral comparisons, that is comparisons in which more than two countries at a time are involved, constitute a subject *sui generis*.

Multilateral international comparisons are not a simple translation of multilateral intertemporal comparisons. Some important differences between both types of comparisons are:

[1] Department of Statistical Methods, Statistics Netherlands, P. O. Box 4000, 2270 JM Voorburg, The Netherlands. Tel 00 31 70 3374704. Telefax 00 31 70 3877429. E-mail bblk@cbs.nl
**Acknowledgement:** This article is a considerably expanded version of a paper presented at the 49th Session of the International Statistical Institute, Firenze, 1993 and at seminars in Karlsruhe and Milano. The author thanks seminar participants, the associate editor, and two anonymous referees for their comments. A special thanks goes to Salem H. Khamis for his very elaborate and insightful comments on the first version. The views expressed in this article are those of the author and do not necessarily reflect the policies of Statistics Netherlands.

- Time proceeds continuously whereas the number of countries involved in a comparison is fixed.
- Unlike time periods, countries do not have a natural order.
- In an intertemporal comparison, the time periods considered are usually of the same size (one compares months with months, years with years, etc.). Countries, however, are by nature not equally "important" (with respect to area, population, economic potential, etc.).
- More than in intertemporal comparisons, there is in international comparisons a strong desire to aggregate the entities and use such aggregates also in comparisons (e.g., the European countries compared to the European Community as a whole, the European Community compared to the United States).

In his recent dissertation, Hill (1995) developed an interesting and virtually complete[2] taxonomy of multilateral methods for international comparisons. This taxonomy provides insight into the structural similarities and dissimilarities of the various methods. However, a taxonomy as such is not enough to discriminate between competing methods. In addition we need a set of criteria, in the spirit of Fisher (1922) called tests, which a multilateral method should satisfy. Such a set of tests has been proposed by Diewert (1986). In the present article we study the performance of a number of multilateral methods with respect to these tests.

This article unfolds as follows. After some introductory material, Section 2 opens with one of the oldest multilateral methods, namely the "balanced" method of Van IJzeren. Closely related to this method are the so-called YKS and Q-YKS methods. We then turn to the weighted EKS method and the weighted Own Share system. The fourth part of Section 2 introduces a number of related methods, namely those of Geary-Khamis, Iklé, Gerardi, and the so-called KS-S method. Section 2 closes with a method proposed by Rao. Section 3 discusses the system of tests, invokes some results by Diewert (1986) and Balk (1989), and presents a number of new results. In the concluding Section 4 we summarize the evidence obtained so far, and discuss some issues which are not covered by the tests. Section 5 contains the technical proofs.

## 2. Multilateral Methods

### 2.1. Notation and general definitions

The following notation is used. There are $I$ countries, $i, j, k = 1, \ldots, I$, which we wish to compare with respect to a certain economic aggregate. This aggregate consists of $N$ commodities (goods and/or services), $n = 1, \ldots, N$. The classification is identical for all countries. Ideally each commodity is a sufficiently homogeneous class of transactions, so that it has a price (unit value) and a (net) quantity. The price vector for country $i$ expressed in its own currency is denoted by $\mathbf{p}^i \equiv (p_1^i, \ldots, p_N^i) \in R_{++}^N$ for $i = 1, \ldots, I$. The corresponding quantity vector is $\mathbf{x}^i \equiv (x_1^i, \ldots, x_N^i) \in R^N$. We allow for negative quantities, which can occur if the aggregate under study is a balance

[2] Hill's taxonomy does not include the Best Linear Index approach (see Theil 1960) or the Factorial Approach (see Rao and Banerjee 1986).

of outputs and inputs. However, we assume that the inner products $\mathbf{p}^i \cdot \mathbf{x}^j \equiv \Sigma_{n=1}^N p_n^i x_n^j > 0$ for all $i, j = 1, \ldots, I$.[3]

The value of country $i$'s aggregate, in its own currency, is $\mathbf{p}^i \cdot \mathbf{x}^i$ $(i = 1, \ldots, I)$. As such, these aggregates are incomparable, because they are expressed in different currencies. A set of purchasing power parities is an instrument that can be used to convert values expressed in different currencies into a common currency. The converted values are then comparable and can, for instance, be added. Suppose we have a set of positive (scalar) purchasing power parities $P^1, \ldots, P^I$, determined up to a scalar multiple. Then $\mathbf{p}^i \cdot \mathbf{x}^i / P^i$ is the converted value of country $i$'s aggregate. We call it country $i$'s volume and we call

$$Q^i \equiv (\mathbf{p}^i \cdot \mathbf{x}^i / P^i) \Bigg/ \left( \sum_{j=1}^{I} \mathbf{p}^j \cdot \mathbf{x}^j / P^j \right)$$

$$= \left( \sum_{j=1}^{I} (\mathbf{p}^j \cdot \mathbf{x}^j / \mathbf{p}^i \cdot \mathbf{x}^i)(P^j / P^i)^{-1} \right)^{-1} \qquad (1)$$

the volume share of country $i$ in the aggregate volume of all $I$ countries. Notice that $\Sigma_{i=1}^I Q^i = 1$. The volume index of country $i$ relative to country $j$ is given by

$$Q^i / Q^j = (\mathbf{p}^i \cdot \mathbf{x}^i / \mathbf{p}^j \cdot \mathbf{x}^j) / (P^i / P^j). \qquad (2)$$

The ratio $P^i / P^j$ is called the purchasing power parity of country $i$ relative to country $j$.

Notice that by our definition, volume indices and purchasing power parities are transitive. This implies that the country chosen as base country in a tabulation of index numbers only acts as a numéraire. Notice further that expression (2) is the multilateral analog of the Product Test from intertemporal index theory: The purchasing power parity times the volume index equals the value ratio. These are the maintained properties of all the methods discussed in this article.

### 2.2. *The Van IJzeren and related methods*

Van IJzeren developed three methods for calculating purchasing power parities and volume indices. They are extensively documented in Van IJzeren (1956), (1983), (1987) and (1988). Following Balk (1989), his "balanced" method, being the most important of the three, can be introduced as follows.

To start with, we define vectors of "international" prices and quantities as

$$\pi \equiv \sum_{i=1}^{I} g^i (\mathbf{p}^i / P^i) \Bigg/ \sum_{i=1}^{I} g^i \qquad (3)$$

$$\chi \equiv \sum_{i=1}^{I} g^i (\mathbf{x}^i / Q^i) \Bigg/ \sum_{i=1}^{I} g^i \qquad (4)$$

where $g^i$ $(i = 1, \ldots, I)$ are positive scalar country weights. These weights reflect the fact that the countries are not equally "important." Expression (3) states that the

[3] This implies that with respect to the aggregate under study the countries must not form a closed system, since then $\Sigma_{j=1}^I \mathbf{x}^j = 0$, which could contradict the assumption.

"international" price of commodity $n$, $\pi_n$, is a weighted arithmetic average of the converted country prices $p_n^i/P^i$. The symmetrical expression (4) means that the "international" quantity of commodity $n$, $\chi_n$, is a weighted arithmetic average of the converted country quantities $x_n^i/Q^i$. Alternatively, the international prices and quantities can be interpreted as the prices and quantities of a fictitious country that serves as a base for the comparisons. Using the international quantities (prices) the price level (quantity level) of country $j$ is defined as $\mathbf{p}^j \cdot \chi$ $(\pi \cdot \mathbf{x}^j)$. These levels are determined up to a scalar factor.

The second step is to link the price levels $\mathbf{p}^j \cdot \chi$ with the purchasing power parities $P^j$ and the quantity levels $\pi \cdot \mathbf{x}^j$ with the volume shares $Q^j$, such that

$$\mathbf{p}^j \cdot \chi = c^j P^j \qquad (j = 1, \ldots, I) \tag{5a}$$

$$\pi \cdot \mathbf{x}^j = c^j Q^j \tag{5b}$$

for certain scalar factors $c^1, \ldots, c^I$. Combining both equations we obtain

$$\mathbf{p}^j \cdot \chi / P^j = \pi \cdot \mathbf{x}^j / Q^j \qquad (j = 1, \ldots, I). \tag{6}$$

Van IJzeren called (6) the "balancing principle": the value of international quantities in country $j$'s converted prices equals the value of country $j$'s converted quantities in international prices. Substituting (2), (3) and (4) into (6) we obtain

$$\sum_{i=1}^{I} g^i P^{i\,j}_{\;L} P^i / P^j = \sum_{i=1}^{I} g^i P^{j\,i}_{\;L} P^j / P^i \qquad (j = 1, \ldots, I), \tag{7}$$

where

$$P^{i\,j}_{\;L} \equiv \mathbf{p}^j \cdot \mathbf{x}^i / \mathbf{p}^i \cdot \mathbf{x}^i \tag{8}$$

is the Laspeyres price index of country $j$ relative to country $i$. The system of equations (7) has a unique positive solution $P^1, \ldots, P^I$ (up to a scalar multiple), as was proved by Van IJzeren (1956). It is easy to show that for $I = 2$ the solution is

$$P^2/P^1 = P^{1\,2}_{\;F} \equiv ((\mathbf{p}^2 \cdot \mathbf{x}^1 / \mathbf{p}^1 \cdot \mathbf{x}^1)(\mathbf{p}^2 \cdot \mathbf{x}^2 / \mathbf{p}^1 \cdot \mathbf{x}^2))^{1/2} \tag{9}$$

that is, the Fisher (ideal) price index of country 2 relative to country 1.

If we substitute the following relation into (7)

$$P^{i\,j}_{\;L} = (\mathbf{p}^j \cdot \mathbf{x}^j / \mathbf{p}^i \cdot \mathbf{x}^i) Q^{j\,i}_{\;L} \tag{10}$$

where

$$Q^{i\,j}_{\;L} \equiv \mathbf{p}^i \cdot \mathbf{x}^j / \mathbf{p}^i \cdot \mathbf{x}^i \tag{11}$$

is the Laspeyres quantity index of country $j$ relative to country $i$, and use (2) we obtain the dual system of equations

$$\sum_{i=1}^{I} g^i Q^{i\,j}_{\;L} Q^i / Q^j = \sum_{i=1}^{I} g^i Q^{j\,i}_{\;L} Q^j / Q^i \qquad (j = 1, \ldots, I). \tag{12}$$

Notice that we can also obtain (12) from (7) by interchanging prices and quantities and replacing purchasing power parities by volume shares. Thus Van IJzeren's $P^i$

and $Q^i$ satisfy a generalized version of the Factor Reversal Test, familiar from intertemporal index theory.

Until now we assumed that the country weights $g^1, \dots, g^I$ were exogenously determined. It appears, however, that this is not necessary. The equality (12) implies that for every vector of normalized country weights $(g^1/\Sigma_{i=1}^I g^i, \dots, g^I/\Sigma_{i=1}^I g^i)$ we obtain a vector of volume shares $(Q^1, \dots, Q^I)$. Thus the system of equations (12) defines a (continuous) mapping from the $I$-dimensional unit simplex into itself. According to Brouwer's Fixed Point Theorem (see Green and Heller 1981) this mapping has a fixed point $g^i/\Sigma_{i=1}^I g^i = Q^i$ $(i = 1, \dots, I)$. The corresponding system of equations is

$$\sum_{i=1}^{I} Q^i\,{}^j_L(Q^i)^2/Q^j = \sum_{i=1}^{I} Q^j\,{}^i_L Q^j \qquad (j = 1, \dots, I). \tag{13}$$

Closely related to the Van IJzeren method, defined by (7), is the YKS method (Kurabayashi and Sakuma 1982, 1990), which is defined by the following system of equations

$$\sum_{i=1}^{I} g^i P^i\,{}^j_L = \sum_{i=1}^{I} g^i P^j\,{}^i_L (P^j/P^i)^2 \quad (j = 1, \dots, I). \tag{14}$$

This method employs differently defined "international" prices and quantities. Replace (3) and (4) by

$$\pi \equiv \sum_{i=1}^{I} g^i(\mathbf{p}^i/(P^i)^2) \Big/ \sum_{i=1}^{I} g^i \tag{15}$$

$$\chi \equiv \sum_{i=1}^{I} g^i(\mathbf{x}^i/P^i Q^i) \Big/ \sum_{i=1}^{I} g^i, \tag{16}$$

substitute into (6), and use (2). Then one obtains expression (14). The existence and uniqueness of the solution can be proved as follows. Let $m^{ij} \equiv g^i P^j\,{}^i_L / \Sigma_{k=1}^I g^k P^k\,{}^j_L$ $(i,j = 1, \dots, I)$. Rewriting (14) we obtain

$$1/(P^j)^2 = \sum_{i=1}^{I} m^{ij}(1/(P^i)^2) \qquad (j = 1, \dots, I). \tag{17}$$

Thus $(1/(P^1)^2, \dots, 1/(P^I)^2)$ is a (left) eigenvector of the $I \times I$ matrix $\mathbf{M}$ with elements $m^{ij}$. Since all elements of $\mathbf{M}$ are positive, $\mathbf{M}$ has a unique positive eigenvector corresponding to the (in absolute value) largest eigenvalue, which is real and positive. Consider the positive vector $\boldsymbol{\sigma}$ with elements $\sigma^j \equiv g^j \Sigma_{k=1}^I g^k P^k\,{}^j_L$ $(j = 1, \dots, I)$. One verifies easily that

$$\mathbf{M}\boldsymbol{\sigma}^T = \boldsymbol{\sigma}^T \tag{18}$$

where $T$ denotes transposition. This implies that the largest eigenvalue of $\mathbf{M}$ is equal to 1 (Theorem of Perron-Frobenius; see Takayama 1974, p. 372). Thus (14) has a unique (up to a positive scalar factor) positive solution $P^1, \dots, P^I$.

The Q-YKS method (Kurabayashi and Sakuma 1982, 1990) is defined by the following system of equations (cf. (12))

$$\sum_{i=1}^{I} g^i Q^{i\,j}_{\;L} = \sum_{i=1}^{I} g^i Q^{j\,i}_{\;L} (Q^j/Q^i)^2 \qquad (j = 1, \ldots, I). \tag{19}$$

We obtain this system of equations by defining

$$\pi \equiv \sum_{i=1}^{I} g^i (\mathbf{p}^i / P^i Q^i) \Big/ \sum_{i=1}^{I} g^i \tag{20}$$

$$\chi \equiv \sum_{i=1}^{I} g^i (\mathbf{x}^i / (Q^i)^2) \Big/ \sum_{i=1}^{I} g^i, \tag{21}$$

substituting this into (6), and using (2). The existence and uniqueness of the solution of (19) are proved in the same way as above.

It is clear that (19) is not a dual representation of (14). For $I > 2$, the YKS method and the Q-YKS method lead to different purchasing power parities and volume shares. For $I = 2$ the solution of (14) is

$$P^2/P^1 = P^{1\,2}_{\;F} \tag{22}$$

and the solution of (19) is

$$Q^2/Q^1 = Q^{1\,2}_{\;F} \equiv ((\mathbf{p}^1 \cdot \mathbf{x}^2 / \mathbf{p}^1 \cdot \mathbf{x}^1)(\mathbf{p}^2 \cdot \mathbf{x}^2 / \mathbf{p}^2 \cdot \mathbf{x}^1))^{1/2} \tag{23}$$

that is, the Fisher (ideal) quantity index of country 2 relative to country 1. However, by relation (2), (23) implies (22) and vice versa. Thus for $I = 2$ the YKS method and the Q-YKS method coincide.

### 2.3. *The EKS method and the Own Share system*

We now consider the relationship between the purchasing power parities according to the Van IJzeren, YKS and Q-YKS methods on the one hand and those according to the EKS method and the Own Share system on the other hand.

The lefthand and righthand side of expression (7), after dividing both sides by $\Sigma_{i=1}^{I} g^i$, can be considered as weighted arithmetic averages. Let us replace the arithmetic averages by geometric averages. This yields

$$\prod_{i=1}^{I} (P^{i\,j}_{\;L} P^i / P^j)^{f^i} = \prod_{i=1}^{I} (P^{j\,i}_{\;L} P^j / P^i)^{f^i} \qquad (j = 1, \ldots, I) \tag{24}$$

where $f^i \equiv g^i / \Sigma_{i=1}^{I} g^i$ $(i = 1, \ldots, I)$. Rewriting this, we obtain

$$\prod_{i=1}^{I} (P^j / P^i)^{f^i} = \prod_{i=1}^{I} (P^{i\,j}_{\;F})^{f^i} \qquad (j = 1, \ldots, I). \tag{25}$$

Dividing (25) by the analogous expression for $j' \neq j$, and using the fact that $\Sigma_{i=1}^{I} f^i = 1$, we obtain for the purchasing power parity of country $j$ relative to country $j'$

$$\begin{aligned} P^j/P^{j'} &= \prod_{i=1}^{I} (P^{i\,j}_{\ F}/P^{i\,j'}_{\ F})^{f^i} \\ &= \prod_{i=1}^{I} (P^{j'\,i}_{\ F} P^{i\,j}_{\ F})^{f^i} \qquad (j, j' = 1, \ldots, I;\ j \neq j'). \end{aligned} \tag{26}$$

This is the weighted version of the EKS purchasing power parity. In its unweighted form (that is, with all $f^i = 1$), expression (26) was already proposed by Gini (1924, p. 110), but independently rediscovered by Eltetö and Köves (1964) and Szulc (1964).[4] Since its introduction into the English speaking world by Drechsler (1973) it became known as the EKS purchasing power parity. More precisely, we should call (26) the (weighted) EKS-Fisher purchasing power parity, because the EKS method of transitivizing intransitive indices can be applied to all bilateral indices which satisfy $P^{ij} = 1/P^{ji}$.[5] Unweighted EKS-Fisher purchasing power parities are currently used by Eurostat in the European Community comparison.

If we apply the same transformation – replacing arithmetic averages by geometric averages – to expression (14), which defines the YKS purchasing power parities, we also obtain (26). If we apply it to expression (19), which defines the Q-YKS volume shares (after normalization), we obtain for the volume index of country $j$ relative to country $j'$

$$Q^j/Q^{j'} = \prod_{i=1}^{I} (Q^{j'\,i}_{\ F} Q^{i\,j}_{\ F})^{f^i} \qquad (j, j' = 1, \ldots, I;\ j \neq j'). \tag{27}$$

However, using relation (2) it is easy to show that (27) is equivalent to (26). Thus the EKS-Fisher purchasing power parity and the EKS-Fisher volume index satisfy the Factor Reversal Test. Notice that for $I = 2$ expression (26) boils down to the bilateral Fisher price index (22) and expression (27) boils down to the bilateral Fisher quantity index (23). Using (1) and (2) we obtain for the EKS-Fisher volume shares the following expression

$$Q^i_{\mathrm{EKS}} \equiv \prod_{k=1}^{I} (Q^{k\,i}_{\ F})^{f^k} \bigg/ \sum_{j=1}^{I} \left( \prod_{k=1}^{I} (Q^{k\,j}_{\ F})^{f^k} \right) \qquad (i = 1, \ldots, I). \tag{28}$$

When we now replace in (28) geometric averages by harmonic averages, we

[4] An alternative derivation of (26) proceeds as follows: Calculate all bilateral index numbers $P^{i\,j}_{\ F}$, rescale each row of this matrix to the common numéraire $j'$, and take the weighted geometric average of these rescaled index numbers per column.

[5] If Kloek and Theil (1965) had not applied a certain constraint, they would have discovered the unweighted EKS-Törnqvist purchasing power parity. The latter index was proposed by Caves, Christensen and Diewert (1982).

obtain

$$Q^i_D \equiv \left(\sum_{k=1}^{I} f^k (Q^{k\ i}_{\ F})^{-1}\right)^{-1} \Big/ \sum_{j=1}^{I}\left(\sum_{k=1}^{I} f^k (Q^{k\ j}_{\ F})^{-1}\right)^{-1}$$
$$= \left(\sum_{k=1}^{I} g^k Q^{i\ k}_{\ F}\right)^{-1} \Big/ \sum_{j=1}^{I}\left(\sum_{k=1}^{I} g^k Q^{j\ k}_{\ F}\right)^{-1}$$
$$= \left(\sum_{j=1}^{I}\left(\sum_{k=1}^{I} g^k Q^{i\ k}_{\ F} \Big/ \sum_{k=1}^{I} g^k Q^{j\ k}_{\ F}\right)\right)^{-1} \qquad (i = 1, \ldots, I) \tag{29}$$

where we used the definition of $f^i$ $(i = 1, \ldots, I)$. This is the weighted version of the Own Share-Fisher system proposed by Diewert (1986). Notice that for $I = 2$ we obtain

$$Q^2_D/Q^1_D = (g^1 + g^2 Q^{1\ 2}_{\ F})/(g^1 Q^{2\ 1}_{\ F} + g^2) = Q^{1\ 2}_{\ F} \tag{30}$$

that is, again the bilateral Fisher quantity index.

### 2.4. *The Geary-Khamis and related methods*

A prominent role in international comparisons is played by the Geary (1958)–Khamis (1972) method (see United Nations 1992; the various phases of the International Comparison Programme are documented in Kravis, Kenessey, Heston, and Summers 1975, Kravis, Heston, and Summers 1978, 1982, and United Nations 1986). Additional material on the method can be found in Khamis (1984). The purchasing power parities according to the GK method are found by solving the following system of equations

$$\pi_n = \left(\sum_{i=1}^{I} p^i_n x^i_n / P^i\right) \Big/ \sum_{i=1}^{I} x^i_n \qquad (n = 1, \ldots, N) \tag{31a}$$

$$P^i = \mathbf{p}^i \cdot \mathbf{x}^i / \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I). \tag{31b}$$

Notice that the purchasing power parity $P^i$ has the form of a Paasche price index of country $i$ relative to the "country" with "international" prices $\boldsymbol{\pi}$. Notice further that (31a) resembles (3) but is not a special case of it: The scalar country weights $g^i$ in (3) are replaced by vectors with elements $x^i_n$. We can consider $\pi_n$ as the unit value of commodity $n$ after converting all values to a common currency.[6]

If we substitute (1) into (31a, b) we obtain the following, equivalent system of equations

$$\pi_n = \alpha \sum_{i=1}^{I} w^i_n Q^i \Big/ \sum_{i=1}^{I} x^i_n \qquad (n = 1, \ldots, N) \tag{32a}$$

$$\alpha Q^i = \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I) \tag{32b}$$

[6] In practice, one chooses a numéraire country, say the first one, and in (31a, b) substitute $\pi_n$ with $\pi_n/p^1_n$, $x^i_n$ with $p^i_n x^i_n/(p^i_n/p^1_n)$, and $p^i_n$ with $p^i_n/p^1_n$, and then solve for $\pi_n/p^1_n$ and $P^i$. Thus only data on values and price relatives are needed.

where $w_n^i \equiv p_n^i x_n^i / \mathbf{p}^i \cdot \mathbf{x}^i$ ($i = 1, \ldots, I; n = 1, \ldots, N$) and $\alpha$ is a certain scalar factor. Substituting (32a) into (32b) we obtain

$$\sum_{i=1}^{I} \left( \sum_{n=1}^{N} \left( w_n^i x_n^j \Big/ \sum_{i=1}^{I} x_n^i \right) \right) Q^i = Q^j \qquad (j = 1, \ldots, I). \tag{33}$$

Let $\mathbf{M}$ be the $I \times I$ matrix with elements $m^{ij} \equiv \Sigma_{n=1}^{N} (w_n^i x_n^j / \Sigma_{i=1}^{I} x_n^i)$. Notice that

$$\sum_{j=1}^{I} m^{ij} = 1 \qquad (i = 1, \ldots, I). \tag{34}$$

Expression (33) states that $(Q^1, \ldots, Q^I)$ is an eigenvector of $\mathbf{M}$. If all elements of $\mathbf{M}$ are positive, $\mathbf{M}$ has a unique positive eigenvector corresponding to the (in absolute value) largest eigenvalue, which is real and positive. Because of (34), this eigenvalue is equal to 1 (Theorem of Perron-Frobenius). Thus if $\mathbf{M}$ is positive (which is certainly the case if all $x_n^i > 0$) the system of equations (31a, b) has a unique (up to a positive scalar factor) positive solution $P^1, \ldots, P^I$.[7]

It is well-known that for $I = 2$ the solution of (31a, b) is

$$P^2 / P^1 = \mathbf{p}^2 \cdot \mathbf{x}^h / \mathbf{p}^1 \cdot \mathbf{x}^h \tag{35}$$

where $\mathbf{x}^h$ is a vector with elements

$$x_n^h \equiv 2 x_n^1 x_n^2 / (x_n^1 + x_n^2) \qquad (n = 1, \ldots, N) \tag{36}$$

that is, harmonic averages of $x_n^1$ and $x_n^2$.

We now consider the following variant of (31a, b), differing only in the definition of the international prices

$$\pi_n = \left( \sum_{i=1}^{I} p_n^i x_n^i / P^i Q^i \right) \Big/ \left( \sum_{i=1}^{I} x_n^i / Q^i \right) \qquad (n = 1, \ldots, N) \tag{37a}$$

$$P^i = \mathbf{p}^i \cdot \mathbf{x}^i / \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I). \tag{37b}$$

This system was proposed by Iklé (1972). The interest in it was revived recently by Dikhanov (1994). Using (1), we obtain an alternative but equivalent formulation

$$\pi_n = \sum_{i=1}^{I} w_n^i \Big/ \sum_{i=1}^{I} w_n^i (p_n^i / P^i)^{-1} \qquad (n = 1, \ldots, N) \tag{38a}$$

$$P^i = \mathbf{p}^i \cdot \mathbf{x}^i / \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I). \tag{38b}$$

Thus we see that each international price $\pi_n$ is a weighted harmonic average of the converted national prices $p_n^i / P^i$. The weights are the country specific value shares $w_n^i$.

Van IJzeren (1983) proved the existence and uniqueness of the solution of (37a, b) for $I = 2$. For the general case we establish the following:

PROPOSITION 1: If all $x_n^i > 0$ ($n = 1, \ldots, N$; $i = 1, \ldots, I$) then the system of

[7] More on necessary and sufficient conditions for the existence of a solution can be found in Khamis (1970), (1972) and Rao (1971).

equations (37a, b) has a unique (up to a positive scalar factor) positive solution $P^1, \ldots, P^I$ (and $Q^1, \ldots, Q^I$).

Proof: Let scalars $q^i > 0$ $(i = 1, \ldots, I)$ such that $\Sigma_{i=1}^{I} q^i = 1$ be given. Then (37a, b) with $q^i$ substituted for $Q^i$ $(i = 1, \ldots, I)$ is isomorphic to the GK system of equations (31a, b) and the condition for the existence of a unique positive solution is satisfied. We denote the solution by $P^1$ $(q^1, \ldots, q^I), \ldots, P^I$ $(q^1, \ldots, q^I)$. Using (1), we obtain the volume shares $Q^1$ $(q^1, \ldots, q^I), \ldots, Q^I$ $(q^1, \ldots, q^I)$, which are positive and add up to 1. The foregoing defines a (continuous) mapping from the $n$-dimensional unit simplex into itself. According to Brouwer's Fixed Point Theorem this mapping has a fixed point $Q^i$ $(q^1, \ldots, q^I) = q^i$ $(i = 1, \ldots, I)$. QED

The next variant of (31a, b) is also obtained by modifying the definition of the international prices. The quantity weights are deleted and instead of arithmetic averages, geometric averages are used. The resulting system of equations reads

$$\pi_n = \left( \prod_{i=1}^{I} p_n^i / P^i \right)^{1/I} \qquad (n = 1, \ldots, N) \tag{39a}$$

$$P^i = \mathbf{p}^i \cdot \mathbf{x}^i / \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I). \tag{39b}$$

However, Khamis, and Rao (1989) demonstrated that this system is inconsistent and admits only the trivial solution $P^i = \pi_n = 0$ $(i = 1, \ldots, I;\ n = 1, \ldots, N)$. Thus instead of (39a, b) we consider

$$\pi_n = \left( \prod_{i=1}^{I} p_n^i \right)^{1/I} \qquad (n = 1, \ldots, N) \tag{40a}$$

$$P^i = \mathbf{p}^i \cdot \mathbf{x}^i / \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I). \tag{40b}$$

This is the Gerardi (1974) method, which was used by Eurostat (1977), (1983) for the European Community 1975 and 1980 comparisons.

Kurabayashi and Sakuma (1981), (1990) introduced a variant of (32a, b), termed the KS-S(implex) method, by the following system of equations

$$\gamma \pi_n = \sum_{i=1}^{I} (p_n^i / \mathbf{p}^i \cdot \boldsymbol{\iota}) Q^i \qquad (n = 1, \ldots, N) \tag{41a}$$

$$\alpha Q^i = \boldsymbol{\pi} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I) \tag{41b}$$

where $\boldsymbol{\iota} \equiv (1, \ldots, 1)$ and $\alpha$ and $\gamma$ are certain scalar factors. Expression (41a) states that each international price $\pi_n$ is proportional to a weighted average of country-specific relative prices $p_n^i / \Sigma_{n=1}^{N} p_n^i$. Substituting (41a) into (41b) we obtain

$$\sum_{i=1}^{I} (\mathbf{p}^i \cdot \mathbf{x}^j / \mathbf{p}^i \cdot \boldsymbol{\iota}) Q^i = \alpha \gamma Q^j \qquad (j = 1, \ldots, I). \tag{42}$$

This implies that $(Q^1, \ldots, Q^I)$ is an eigenvector of the positive matrix with elements $\mathbf{p}^i \cdot \mathbf{x}^j / \mathbf{p}^i \cdot \iota$. Using again the Theorem of Perron-Frobenius we conclude that there exists a unique (up to a positive scalar factor) positive solution $Q^1, \ldots, Q^I$, corresponding to the largest eigenvalue.

Finally we notice a common feature of the methods discussed in this subsection. In all four cases the resulting volume shares can be expressed as

$$Q^i = \boldsymbol{\pi} \cdot \mathbf{x}^i \Big/ \sum_{j=1}^{I} \boldsymbol{\pi} \cdot \mathbf{x}^j \qquad (i = 1, \ldots, I) \tag{43}$$

where $\pi$ is the vector of international prices. The difference between the methods is caused by the different definitions of $\pi$.

### 2.5. *A geometric variant of the Iklé method*

Rao (1990) introduced the following system of equations

$$\pi_n = \prod_{i=1}^{I} (p_n^i / P^i)^{w_n^{i*}} \qquad (n = 1, \ldots, N) \tag{44a}$$

$$P^i = \prod_{n=1}^{N} (p_n^i / \pi_n)^{w_n^i} \qquad (i = 1, \ldots, I) \tag{44b}$$

where $w_n^i \equiv p_n^i x_n^i / \mathbf{p}^i \cdot \mathbf{x}^i$ and $w_n^{i*} \equiv w_n^i / \Sigma_{i=1}^I w_n^i$ $(i = 1, \ldots, I;\ n = 1, \ldots, N)$. An alternative formulation, using logarithms, is

$$\ell n\, \pi_n = \sum_{i=1}^{I} w_n^{i*} \ell n\, (p_n^i / P^i) \qquad (n = 1, \ldots, N) \tag{45a}$$

$$\ell n\, P^i = \sum_{n=1}^{N} w_n^i \ell n\, (p_n^i / \pi_n) \qquad (i = 1, \ldots, I). \tag{45b}$$

Rao compared this so-called system of log-change indices to the Geary-Khamis system (31). However, it is more to the point to consider (44) as the geometric variant of the Iklé method. Notice that (38a) can be written as $\pi_n = (\Sigma_{i=1}^I w_n^{i*} (p_n^i / P^i)^{-1})^{-1}$ $(n = 1, \ldots, N)$ and that (38b) can be written as $P^i = (\Sigma_{n=1}^N w_n^i (p_n^i / \pi_n)^{-1})^{-1}$ $(i = 1, \ldots, I)$. The analogy between (38a) and (44a) and between (38b) and (44b) is now clear: interchange harmonic and geometric averages.

We now consider whether this system has a solution. When we substitute (45a) into (45b) and rearrange terms, we obtain the following matrix equation

$$(\mathbf{I} - \mathbf{A})\, \ell n\, \mathbf{P}^T = \mathbf{b}^T \tag{46}$$

where $\mathbf{I}$ is the $I \times I$ unit matrix, $\mathbf{A}$ is the $I \times I$ matrix with elements $a_{ij} \equiv \Sigma_{n=1}^N w_n^i w_n^{j*}$, $\ell n\, \mathbf{P} \equiv (\ell n\, P^1, \ldots, \ell n\, P^I)$, $\mathbf{b}$ is the $1 \times I$ vector with elements $b_i \equiv \Sigma_{n=1}^N \Sigma_{j=1}^I w_n^i w_n^{j*} \times (\ell n\, p_n^i - \ell n\, p_n^j)$, and $T$ is the transposition symbol. One verifies directly that $\mathbf{A}$ is symmetric and that

$$\iota \mathbf{A} = \iota \text{ and } \iota \mathbf{b}^T = 0. \tag{47}$$

These properties imply that if $(P^1, \ldots, P^I)$ is a solution of (46), then also $(\alpha P^1, \ldots, \alpha P^I)$ for any $\alpha > 0$. In particular, we can choose $\alpha = 1/P^I$. It is then straightforward to show that (46) is equivalent to

$$(\tilde{\mathbf{I}} - \tilde{\mathbf{A}})\boldsymbol{\pi}^T = \tilde{\mathbf{b}}^T \tag{48}$$

where $\tilde{\mathbf{I}} - \tilde{\mathbf{A}}$ is $\mathbf{I} - \mathbf{A}$ with the $I$th row and column deleted, $\boldsymbol{\pi} \equiv (\ell\text{n}\,(P^1/P^I), \ldots, \ell\text{n}\,(P^{I-1}/P^I))$, and $\tilde{\mathbf{b}} \equiv (b_1, \ldots, b_{I-1})$. Assume that all $a_{ij}$ are positive, and consider the matrix $\tilde{\mathbf{I}} - \tilde{\mathbf{A}}$. Its diagonal elements are $1 - a_{ii}$, its off-diagonal elements are $-a_{ij}$, and

$$1 - a_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{I-1} a_{ij} \qquad (i = 1, \ldots, I-1). \tag{49}$$

Thus $\tilde{\mathbf{I}} - \tilde{\mathbf{A}}$ has a dominant diagonal. The Theorem of Hadamard-McKenzie (see Takayama 1974, p. 381) then implies that $\tilde{\mathbf{I}} - \tilde{\mathbf{A}}$ is non-singular. Hence,

$$\boldsymbol{\pi}^T = (\tilde{\mathbf{I}} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{b}}^T. \tag{50}$$

Thus if all $a_{ij}$ are positive, the Rao system of equations (44a, b) has a unique (up to a positive scalar factor) positive solution $P^1, \ldots, P^I$.

## 3. Comparison of the Methods with Respect to Diewert's Tests

### 3.1. *The set of tests*

Diewert (1986) formulated a set of tests for multilateral comparisons; cf. Diewert (1987) for a rephrased version. Balk (1989) modified the tests by incorporating country weights. The tests emphasize the fact that the primary purpose of any international comparison is to make volume comparisons. Purchasing power parities play only an intermediate role, namely as currency convertors. Consequently, the tests are phrased in terms of volume shares, which are understood to be functions of all prices, all quantities, and all country weights (if any). Formally, for $i = 1, \ldots, I$, $Q^i = Q^i\,(\mathbf{p}^1, \ldots, \mathbf{p}^I, \mathbf{x}^1, \ldots, \mathbf{x}^I, g^1, \ldots, g^I)$.

MT1. Positivity and continuity test. $Q^i > 0$, $i = 1, \ldots, I$; $\Sigma_{i=1}^I Q^i = 1$; the functions $Q^i$ are jointly continuous in $\mathbf{p}^i$, $\mathbf{x}^i$ and $g^i$ $(i = 1, \ldots, I)$.

MT2. Weak proportionality test or identity test. If $\mathbf{p}^j = \alpha^j\mathbf{p}$ and $\mathbf{x}^j = \beta^j\mathbf{x}$ where $\alpha^j, \beta^j > 0$ $(j = 1, \ldots, I)$ and $\Sigma_{j=1}^I \beta^j = 1$, then $Q^i = \beta^i$ for $i = 1, \ldots, I$.

MT3. Proportionality test. Let $\lambda > 0$ and replace for country $k$ the quantity vector $\mathbf{x}^k$ by $\lambda\mathbf{x}^k$ and the weight $g^k$ by $\lambda g^k$. Then the relation between the new volume shares $\tilde{Q}^i$ and the old volume shares $Q^i$ is

$$\begin{aligned} \tilde{Q}^i &= \lambda Q^k/[1 + (\lambda - 1)Q^k] && \text{for } i = k \\ & \quad Q^i/[1 + (\lambda - 1)Q^k] && \text{for } i \neq k. \end{aligned} \tag{51}$$

MT4. Monetary unit test or invariance to changes in scale test. Replace $\mathbf{p}^j$ by $\alpha^j\mathbf{p}^j$

and $\mathbf{x}^j$ by $\beta\mathbf{x}^j$ where $\alpha^j, \beta > 0$ $(j = 1, \ldots, I)$. Then the new volume shares are identically equal to the old volume shares.

MT5. Commensurability test or invariance to changes in units test. The volume shares are invariant with respect to scale changes in the (quantity) units of commodities.

MT6. Symmetric treatment of countries test. The volume shares are invariant with respect to a permutation of the countries.

MT7. Symmetric treatment of commodities test. The volume shares are invariant with respect to a permutation of the commodities.

MT8. Country partitioning test. Let country $k$ be partitioned into two provinces, denoted as $k$ and $I+1$ respectively, with the same price vector $\mathbf{p}^k$ but quantity vectors $\lambda\mathbf{x}^k$ and $(1-\lambda)\mathbf{x}^k$ respectively $(0 < \lambda < 1)$, and let the country weights be $\lambda g^k$ and $(1-\lambda)g^k$ respectively. Then the relation between the new volume shares $\tilde{Q}^i$ and the old volume shares $Q^i$ is

$$\begin{aligned} \tilde{Q}^i &= Q^i \qquad \text{for } i = 1, \ldots, k-1, k+1, \ldots, I \\ \tilde{Q}^{I+1} &= (1-\lambda)Q^k \\ \tilde{Q}^k &= \lambda Q^k. \end{aligned} \tag{52}$$

MT9. Irrelevance of tiny countries test. Let $\lambda > 0$ and replace for country $k$ the quantity vector $\mathbf{x}^k$ by $\lambda\mathbf{x}^k$ and the weight $g^k$ by $\lambda g^k$. Denote the resulting volume shares by $Q^i(\lambda)$ $(i = 1, \ldots, I)$. Delete country $k$ and denote the resulting volume shares by $\tilde{Q}^i$ $(i = 1, \ldots, k-1, k+1, \ldots, I)$. Then

$$\lim_{\lambda \to 0} Q^i(\lambda) = \tilde{Q}^i \qquad (i = 1, \ldots, k-1, k+1, \ldots, I). \tag{53}$$

We add some explanatory remarks. MT2 suggests that if all the price vectors are proportional to each other and all the quantity vectors are also proportional to each other, then the volume shares are equal to the factors of proportionality of the quantity vectors. A specific case is obtained when all price vectors are equal, $\mathbf{p}^1 = \ldots = \mathbf{p}^I$, and all quantity vectors are equal, $\mathbf{x}^1 = \ldots = \mathbf{x}^I$. Then $Q^i = 1/I$ $(i = 1, \ldots, I)$. MT3 suggests that if the prices remain unchanged but country $k$ expands with a certain factor, then the volume shares behave accordingly. MT4 suggests that differing inflation rates but equal quantity growth rates leave the volume shares invariant. MT8 considers the situation in which we wish to disaggregate the countries and require consistency. MT9 stipulates that "small" countries not influence the volume shares of "large" countries unduly.

## 3.2. The results

Balk (1989) showed that the Van IJzeren volume shares, defined by (1) and (7), fail to satisfy only MT3. Concerning the YKS volume shares and the Q-YKS volume shares we establish the following results.

PROPOSITION 2. The YKS volume shares, defined by (1) and (14), fail to satisfy only MT3 and MT4.

PROPOSITION 3. The Q-YKS volume shares, defined by (19), fail to satisfy only MT8 and MT9.

By tedious but straightforward calculation, one can prove (cf. for the unweighted case Diewert 1986, Proposition 8):

PROPOSITION 4. The volume shares corresponding to the weighted EKS-Fisher method (26) and those corresponding to the weighted Own Share-Fisher system (29) fail to satisfy only MT3 and MT8.

Diewert (1986, Proposition 13) showed that the volume shares corresponding to the Geary-Khamis method (31) fail to satisfy only MT3.

PROPOSITION 5. The volume shares corresponding to the Iklé method (37) fail to satisfy only MT8 and MT9.

PROPOSITION 6. The volume shares corresponding to the Gerardi method (40) fail to satisfy only MT8 and MT9.

PROPOSITION 7. The volume shares corresponding to the KS-S method (41) fail to satisfy only MT3 and MT5.

PROPOSITION 8. The volume shares corresponding to the Rao method (44) fail to satisfy only MT8 and MT9.

## 4. Concluding Remarks

The results obtained in the foregoing section can be summarized in Table 1. It appears that there is no method which satisfies all the tests. In particular, the simultaneous satisfaction of MT3 and MT8/MT9 seems to be impossible. Whether the system of tests is inconsistent, however, is as yet an open question. We can conclude that, with respect to the number of tests failed, the Van IJzeren method and the Geary-Khamis method show superiority to all the other methods. Both methods fail to

*Table 1. Performance of the various methods with respect to Diewert's tests*

| Method | Defining expression | Tests failed |
|---|---|---|
| Van IJzeren | (7) | MT3 |
| YKS | (14) | MT3, MT4 |
| Q-YKS | (19) | MT8, MT9 |
| EKS-Fisher | (26) | MT3, MT8 |
| Own Share | (29) | MT3, MT8 |
| Geary-Khamis | (31) | MT3 |
| Iklé | (37) | MT8, MT9 |
| Gerardi | (40) | MT8, MT9 |
| KS-S | (41) | MT3, MT5 |
| Rao | (44) | MT8, MT9 |

satisfy only MT3, the proportionality test. We therefore look at the relative performance of both methods with respect to this test.

Following Geary (1958) and Diewert (1986) we replace the country $k$ quantity vector $\mathbf{x}^k$ by $\lambda\mathbf{x}^k$ $(\lambda > 0)$ in the equations (31a, b) and let $\lambda$ tend to infinity. Then the Geary-Khamis volume indices and purchasing power parities for country $i$ relative to country $j$ behave as follows

$$Q^i/Q^j \rightarrow \mathbf{p}^k{\cdot}\mathbf{x}^i/\mathbf{p}^k{\cdot}\mathbf{x}^j = Q^{k\,i}_{\;L}/Q^{k\,j}_{\;L} \qquad (i,j = 1,\ldots,I) \tag{54a}$$

$$P^i/P^j \rightarrow P^{k\,i}_{\;P}/P^{k\,j}_{\;P} \qquad (i,j = 1,\ldots,I) \tag{54b}$$

where $P^{k\,i}_{\;P} \equiv \mathbf{p}^i{\cdot}\mathbf{x}^i/\mathbf{p}^k{\cdot}\mathbf{x}^i$ is the Paasche price index of country $i$ relative to country $k$. Thus the limiting purchasing power parity is a ratio of bilateral Paasche price indices, and the limiting volume index is a ratio of bilateral Laspeyres quantity indices.

We now turn to the Van IJzeren method. Replacing $\mathbf{x}^k$ by $\lambda\mathbf{x}^k$ and $g^k$ by $\lambda g^k$ $(\lambda > 0)$ and taking limits as $\lambda$ tends to infinity, we obtain from (7)

$$P^{k\,j}_{\;L}P^k/P^j = P^{j\,k}_{\;L}P^j/P^k \qquad (j = 1,\ldots,I). \tag{55}$$

The (positive) solution is

$$P^j/P^k = P^{k\,j}_{\;F} \qquad (j = 1,\ldots,I). \tag{56}$$

This implies

$$P^i/P^j = P^{k\,i}_{\;F}/P^{k\,j}_{\;F} \qquad (i,j = 1,\ldots,I) \tag{57a}$$

$$Q^i/Q^j = Q^{k\,i}_{\;F}/Q^{k\,j}_{\;F} \qquad (i,j = 1,\ldots,I). \tag{57b}$$

The limiting purchasing power parity is now a ratio of bilateral Fisher price indices, and the limiting volume index is a ratio of bilateral Fisher quantity indices. Diewert (1986) showed that from a microeconomic perspective the bilateral Fisher index is preferable to the Paasche or the Laspeyres index. This view, however, is not endorsed by Khamis (1984). He writes on page 194: "The author is of the opinion that the persistent use of the Fisher index by many practitioners and official agencies is one of the most serious drawbacks in the history of index numbers and their application."

The underlying issue seems to be the importance one attaches to additivity or, as Kurabayashi and Sakuma (1990) call it, "matrix consistency." A distinctive feature of the Geary-Khamis method (and the other methods discussed in Section 2.4) is its additivity, which is reflected in expression (32b). We restate this expression here

$$\alpha Q^i = \boldsymbol{\pi}{\cdot}\mathbf{x}^i \qquad (i = 1,\ldots,I). \tag{32b}$$

Additivity implies that

$$Q^i/Q^j = \boldsymbol{\pi}{\cdot}\mathbf{x}^i/\boldsymbol{\pi}{\cdot}\mathbf{x}^j = \sum_{n=1}^{N}(\pi_n x_n^j/\boldsymbol{\pi}{\cdot}\mathbf{x}^j)(x_n^i/x_n^j) \qquad (i,j = 1,\ldots,I) \tag{58}$$

that is, the volume index is a weighted average of the quantity ratios. Otherwise stated, the volume index satisfies the so-called Average Test, which states that an index should lie between the smallest and the largest of the single relatives. Notice however that the associated purchasing power parity may violate this test (see

Drechsler and Krzeczkowska 1982). On the other hand the Van IJzeren method is characterized by the fact that

$$c^i Q^i = \boldsymbol{\pi}\cdot\mathbf{x}^i \qquad (i = 1, \ldots, I) \tag{5b}$$

which implies that

$$Q^i/Q^j = (\boldsymbol{\pi}\cdot\mathbf{x}^i/\boldsymbol{\pi}\cdot\mathbf{x}^j)(c^j/c^i) \qquad (i, j = 1, \ldots, I). \tag{59}$$

This volume index, as well as the corresponding purchasing power parity, may violate the Average Test. In a sense, the Van IJzeren method implies instead of additivity a kind of non-linear aggregation, which is reflected by the scalar factors $c^i$ $(i = 1, \ldots, I)$. For a further discussion of these factors we refer to Balk (1989).

For many purposes, additivity is a desirable feature. However, the pleasure of additivity is bought at the price of suffering from the so-called Gerschenkron effect. This effect is caused by a phenomenon familiar from intertemporal comparisons: the negative correlation between relative price differences and relative quantity differences in a consumer dominated market or the positive correlation in a producer dominated market. In intertemporal theory we speak about substitution behavior, and explain it by using the model of an optimizing agent.

Assuming optimizing behavior on the part of economic agents, Diewert (1995) showed that the linear and the Leontief aggregator functions are the only functions which are exact for an additive volume index of the form (58).[8] The linear function assumes perfect substitutability, while the Leontief function assumes zero substitutability. These are clearly extremely restrictive boundary cases, which are not able to account for the empirical regularities in a satisfactory way. Thus using an additive index is likely to introduce bias.

But also if one does not believe in optimizing behavior, it is generally felt that the Laspeyres index $Q^{k\,i}_{\;\;L}$ over/understates (depending on the character of the market) and the Paasche index $Q^{k\,i}_{\;\;P}$ under/overstates the true volume index $Q^i/Q^k$. In order to avoid such biases, it would therefore be preferable to use the Fisher index $Q^{k\,i}_{\;\;F}$. Referring back to the limiting behavior of the Geary-Khamis indices vis-à-vis the Van IJzeren indices (cf. expressions (54a, b) and (57a, b)) we can conclude that the Geary-Khamis indices run a larger risk of bias than the Van IJzeren indices do. Recent research by Nuxoll (1994) and Hill (1995, chapter 4) confirms this.

Summarizing, we have a preference for the Van IJzeren method over and above the Geary-Khamis method. If one has problems with the Van IJzeren method with respect to its understandability, one could use the easier (weighted) EKS-Fisher method, which provides a numerically close approximation to the Van IJzeren method.[9] Thus the current policy of a number of international organizations, of providing simultaneously Geary-Khamis and EKS-Fisher indices, can be defended from a theoretical perspective.

[8] A related result was obtained by Rao and Salazar-Carrillo (1988).

[9] See the numerical example in Balk (1996). This example was also contained in the first version of the present article and has been quoted from there by Köves (1995).

## 5. Proofs

Proof of Proposition 2: MT3: The new purchasing power parities are the solution of the system

$$\sum_{\substack{i=1\\i\neq k}}^{I} g^i \mathbf{p}^j\cdot\mathbf{x}^i/\mathbf{p}^i\cdot\mathbf{x}^i + \lambda g^k \mathbf{p}^j\cdot\mathbf{x}^k/\mathbf{p}^k\cdot\mathbf{x}^k = \sum_{\substack{i=1\\i\neq k}}^{I} g^i(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^j\cdot\mathbf{x}^j)(\tilde{P}^j/\tilde{P}^i)^2 + \lambda g^k(\mathbf{p}^k\cdot\mathbf{x}^j/\mathbf{p}^j\cdot\mathbf{x}^j)(\tilde{P}^j/\tilde{P}^k)^2 \quad (j=1,\ldots,I).$$

The old purchasing power parities are the solution of the system

$$\sum_{i=1}^{I} g^i \mathbf{p}^j\cdot\mathbf{x}^i/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{i=1}^{I} g^i(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^j\cdot\mathbf{x}^j)(P^j/P^i)^2 \qquad (j=1,\ldots,I). \tag{60}$$

It is clear that for $\lambda \neq 1$ in general $\tilde{P}^i \neq P^i$ $(i=1,\ldots,I)$. Therefore in general (51) will not hold.

MT4: The new purchasing power parities are the solution of the system

$$\sum_{i=1}^{I} g^i(\alpha^j/\alpha^i)\mathbf{p}^j\cdot\mathbf{x}^i/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{i=1}^{I} g^i(\alpha^i/\alpha^j)(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^j\cdot\mathbf{x}^j)(\tilde{P}^j/\tilde{P}^i)^2 \qquad (j=1,\ldots,I).$$

The old purchasing power parities are the solution of the system (60). It is clear that in general $\tilde{P}^i \neq \alpha^i P^i$ $(i=1,\ldots,I)$. This implies that new and old volume shares will differ from each other.

The proof for the remaining tests is by straightforward calculation and exactly parallels the proof of Balk (1989). QED

Proof of Proposition 3: MT1 is satisfied by definition (after normalization such that $\Sigma_{i=1}^{I} Q^i = 1$).

MT2: We obtain the following system of equations

$$\sum_{i=1}^{I} g^i \beta^j/\beta^i = \sum_{i=1}^{I} g^i(\beta^i/\beta^j)(Q^j/Q^i)^2 \qquad (j=1,\ldots,I).$$

The solution is $Q^j/Q^i = \beta^j/\beta^i$ $(i=1,\ldots,I)$.

MT3: The new volume shares $\tilde{Q}^i$ $(i=1,\ldots,I)$ are the solution of the system of equations

$$\sum_{i=1}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{\substack{i=1\\i\neq k}}^{I} g^i(\mathbf{p}^j\cdot\mathbf{x}^i/\mathbf{p}^j\cdot\mathbf{x}^j)(\tilde{Q}^j/\tilde{Q}^i)^2 + \lambda^2 g^k(\mathbf{p}^j\cdot\mathbf{x}^k/\mathbf{p}^j\cdot\mathbf{x}^j)(\tilde{Q}^j/\tilde{Q}^k)^2 \quad (j=1,\ldots,I;\ j\neq k)$$

$$\sum_{i=1}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{\substack{i=1\\i\neq k}}^{I} g^i(\mathbf{p}^k\cdot\mathbf{x}^i/\mathbf{p}^k\cdot\mathbf{x}^k\lambda^2)(\tilde{Q}^k/\tilde{Q}^i)^2 + g^k.$$

If we substitute (51) we obtain the system of equations for the old volume shares.

MT4: It is easy to verify that the new and the old system of equations coincide.

MT5, MT6, and MT7 are satisfied by definition.

MT8: If we write down the system of equations for the $I+1$ countries and substitute (52), we obtain the following system

$$\sum_{\substack{i=1\\ i\neq k}}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\mathbf{x}^i + 2g^k\mathbf{p}^k\cdot\mathbf{x}^j/\mathbf{p}^k\cdot\mathbf{x}^k = \sum_{\substack{i=1\\ i\neq k}}^{I} g^i(\mathbf{p}^j\cdot\mathbf{x}^i/\mathbf{p}^j\cdot\mathbf{x}^j)(Q^j/Q^i)^2 + 2g^k(\mathbf{p}^j\cdot\mathbf{x}^k/\mathbf{p}^j\cdot\mathbf{x}^j)(Q^j/Q^k)^2 \quad (j=1,\ldots,I;\ j\neq k)$$

$$\sum_{i=1}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{i=1}^{I} g^i(\mathbf{p}^k\cdot\mathbf{x}^i/\mathbf{p}^k\cdot\mathbf{x}^k)(Q^k/Q^i)^2.$$

However, this system differs from the system defining the old volume shares.

MT9: The system of equation for the $I$ countries reads

$$\sum_{i=1}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{\substack{i=1\\ i\neq k}}^{I} g^i(\mathbf{p}^j\cdot\mathbf{x}^i/\mathbf{p}^j\cdot\mathbf{x}^j)(Q^j/Q^i)^2 + \lambda^2 g^k(\mathbf{p}^j\cdot\mathbf{x}^k/\mathbf{p}^j\cdot\mathbf{x}^j)(Q^j/Q^k)^2 \quad (j=1,\ldots,I;\ j\neq k) \tag{61a}$$

$$\lambda^2\sum_{\substack{i=1\\ i\neq k}}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\mathbf{x}^i = \sum_{\substack{i=1\\ i\neq k}}^{I} g^i(\mathbf{p}^k\cdot\mathbf{x}^i/\mathbf{p}^k\cdot\mathbf{x}^k)(Q^k/Q^i)^2. \tag{61b}$$

Equation (61b) gives

$$(Q^k)^2 = \lambda^2\left(\sum_{\substack{i=1\\ i\neq k}}^{I} g^i \mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\mathbf{x}^i\right)\Bigg/\left(\sum_{\substack{i=1\\ i\neq k}}^{I} g^i(\mathbf{p}^k\cdot\mathbf{x}^i/\mathbf{p}^k\cdot\mathbf{x}^k)/(Q^i)^2\right). \tag{62}$$

Substituting (62) into (61a) yields a system of $I-1$ equations which does not contain $\lambda$. Thus the latter system cannot converge to the system of equations that we obtain when we delete country $k$. QED

Proof of Proposition 5: MT1 is satisfied by definition (subject to all quantities being positive).

MT2 follows directly by using (43).

MT3: Replacing for country $k$ $\mathbf{x}^k$ by $\lambda\mathbf{x}^k$ leaves the system of equations (38a, b) invariant. Thus the new and the old purchasing power parities are equal to each other, and (51) follows immediately.

MT4: The transformation leaves all value shares $w_n^i$ invariant. The system of equations (38a, b) then implies that $P^i$ transforms into $\alpha^i P^i$. This transformation leaves the volume shares invariant.

MT5, MT6 and MT7 are satisfied by definition.

MT8: The system of equations for the new purchasing power parities is

$$\tilde{\pi}_n = \left[\sum_{\substack{i=1\\i\neq k}}^{I} w_n^i + 2w_n^k\right] \Bigg/ \left[\sum_{\substack{i=1\\i\neq k}}^{I} w_n^i (p_n^i/\tilde{P}^i)^{-1} + 2w_n^k (p_n^k/\tilde{P}^k)^{-1}\right] \quad (n = 1, \ldots, N)$$

$$\tilde{P}^i = \mathbf{p}^i \cdot \mathbf{x}^i / \tilde{\boldsymbol{\pi}} \cdot \mathbf{x}^i \qquad (i = 1, \ldots, I;\ i \neq k).$$

$$\tilde{P}^k = \mathbf{p}^k \cdot \mathbf{x}^k / \tilde{\boldsymbol{\pi}} \cdot \mathbf{x}^k$$

$$\tilde{P}^{I+1} = \tilde{P}^k.$$

It is clear that the solution $\tilde{P}^i$ $(i = 1, \ldots, I)$ of this system differs from the solution $P^i$ $(i = 1, \ldots, I)$ of (38a, b). Hence (52) will not be satisfied.

MT9: The transformation leaves the system of equations (38a, b) invariant. Thus $P^i(\lambda) = P^i(1)$ for $i = 1, \ldots, I$ and

$$Q^i(\lambda) = \lambda Q^k(1)/[1 + (\lambda - 1)Q^k(1)] \qquad \text{for } i = k$$

$$= Q^i(1)/[1 + (\lambda - 1)Q^k(1)] \qquad \text{for } i \neq k.$$

For $\lambda \to 0$

$$Q^k(\lambda) \to 0$$

$$Q^i(\lambda) \to Q^i(1)/[1 - Q^k(1)] \qquad \text{for } i \neq k.$$

However, deleting country $k$ leaves the system (38a, b) not invariant. Thus $\tilde{P}^i$ is not proportional to $P^i(1)$ and $\tilde{Q}^i$ is not proportional to $Q^i(1)$ $(i = 1, \ldots, I;\ i \neq k)$. QED

Proof of Proposition 6: MT1 is satisfied by definition. MT2, MT3 and MT4 follow directly by using (43). MT5, MT6 and MT7 are satisfied by definition.

MT8: The new international prices are

$$\tilde{\pi}_n = \pi_n^{I/(I+1)} (p_n^k)^{1/(I+1)} \qquad (n = 1, \ldots, N)$$

and it is clear that in general $\tilde{\boldsymbol{\pi}}$ is not proportional to $\boldsymbol{\pi}$. Thus (52) will not hold.

MT9: For $i \neq k$

$$Q^i(\lambda) = \boldsymbol{\pi} \cdot \mathbf{x}^i \Bigg/ \left[\sum_{\substack{j=1\\j\neq k}}^{I} \boldsymbol{\pi} \cdot \mathbf{x}^j + \lambda \boldsymbol{\pi} \cdot \mathbf{x}^k\right] \to \boldsymbol{\pi} \cdot \mathbf{x}^i \Bigg/ \sum_{\substack{j=1\\j\neq k}}^{I} \boldsymbol{\pi} \cdot \mathbf{x}^j$$

when $\lambda \to 0$. When country $k$ is deleted, the new international prices are

$$\tilde{\pi}_n = \left(\prod_{\substack{i=1\\i\neq k}}^{I} p_n^i\right)^{1/(I-1)} \qquad (n = 1, \ldots, N)$$

and it is clear that in general $\tilde{\boldsymbol{\pi}}$ is not proportional to $\boldsymbol{\pi}$. Thus the new volume shares

$$\tilde{Q}^i = \tilde{\pi}\cdot\mathbf{x}^i \Big/ \sum_{\substack{j=1\\ j\neq k}}^{I} \tilde{\pi}\cdot\mathbf{x}^j \qquad (i=1,\ldots,I;\ i\neq k)$$

are not proportional to the limiting values of $Q^i(\lambda)$. QED

Proof of Proposition 7: MT1 is satisfied by definition (after normalization).

MT2: The vector $(Q^1,\ldots,Q^I)$ is the unique solution of the system of equations

$$\sum_{i=1}^{I}(\alpha^i\mathbf{p}\cdot\mathbf{x}\beta^j/\alpha^i\mathbf{p}\cdot\iota)Q^i = \delta Q^j \qquad (j=1,\ldots,I).$$

Using the fact that the volume shares sum to 1, this is equivalent to

$$\beta^j\mathbf{p}\cdot\mathbf{x}/\mathbf{p}\cdot\iota = \delta Q^j \qquad (j=1,\ldots,I).$$

Using again the normalization of the $Q^j$ and the $\beta^j$, we find $\mathbf{p}\cdot\mathbf{x}/\mathbf{p}\cdot\iota = \delta$, which implies that $Q^j = \beta^j$ $(j=1,\ldots,I)$.

MT3: The new volume shares are the unique solution of the system of equations

$$\sum_{i=1}^{I}(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\iota)\tilde{Q}^i = \delta\tilde{Q}^j \qquad (j=1,\ldots,I;\ j\neq k)$$

$$\lambda\sum_{i=1}^{I}(\mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\iota)\tilde{Q}^i = \delta\tilde{Q}^k.$$

Substituting (51), we obtain

$$\sum_{\substack{i=1\\ i\neq k}}^{I}(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\iota)Q^i + \lambda(\mathbf{p}^k\cdot\mathbf{x}^j/\mathbf{p}^k\cdot\iota)Q^k = \delta Q^j \qquad (j=1,\ldots,I;\ j\neq k)$$

$$\sum_{\substack{i=1\\ i\neq k}}^{I}(\mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\iota)Q^i + \lambda(\mathbf{p}^k\cdot\mathbf{x}^k/\mathbf{p}^k\cdot\iota)Q^k = \delta Q^k.$$

For $\lambda\neq 1$ the latter system differs from (42).

MT4: The new volume shares are the unique solution of the system of equations

$$\sum_{i=1}^{I}(\alpha^i\mathbf{p}^i\cdot\mathbf{x}^j\beta/\alpha^i\mathbf{p}^i\cdot\iota)\tilde{Q}^i = \delta\tilde{Q}^j \qquad (j=1,\ldots,I).$$

By cancelling factors we obtain

$$\sum_{i=1}^{I}(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\iota)\tilde{Q}^i = (\delta/\beta)\tilde{Q}^j \qquad (j=1,\ldots,I).$$

This is the same system as (42). Thus $\tilde{Q}^i = Q^i$ for $i=1,\ldots,I$.

MT5 is not satisfied since $\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\iota$ is not invariant with respect to scale changes.

MT6 and MT7 are satisfied by definition.

MT8: If we write down the system of equations for the new volume shares and substitute (52), we obtain the system for the old volume shares.

MT9: The volume shares $Q^1(\lambda), \ldots, Q^I(\lambda)$ are the unique solution of the system

$$\sum_{\substack{i=1\\i\neq k}}^{I}(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\iota)Q^i(\lambda) + (\mathbf{p}^k\cdot\mathbf{x}^j/\mathbf{p}^k\cdot\iota)Q^k(\lambda) = \delta Q^j(\lambda) \qquad (j=1,\ldots,I;\ j\neq k)$$

$$\lambda\sum_{\substack{i=1\\i\neq k}}^{I}(\mathbf{p}^i\cdot\mathbf{x}^k/\mathbf{p}^i\cdot\iota)Q^i(\lambda) + \lambda(\mathbf{p}^k\cdot\mathbf{x}^k/\mathbf{p}^k\cdot\iota)Q^k(\lambda) = \delta Q^k(\lambda).$$

The last equation yields an expression for $Q^k(\lambda)$ which converges to 0 when $\lambda \to 0$. Then the first $n-1$ equations converge to

$$\sum_{\substack{i=1\\i\neq k}}^{I}(\mathbf{p}^i\cdot\mathbf{x}^j/\mathbf{p}^i\cdot\iota)Q^i(0) = \delta Q^j(0) \qquad (j=1,\ldots,I;\ j\neq k).$$

Thus $Q^i(0) = \tilde{Q}^i$ $(j=1,\ldots,I;\ j\neq k)$. QED

Proof of Proposition 8: MT1 is satisfied by definition (subject to the existence condition).

MT2: Substitute $\mathbf{p}^i = \alpha^i\mathbf{p}$ and $\mathbf{x}^i = \beta^i\mathbf{x}$ $(i=1,\ldots,I)$ into the system of equations (44a, b). The solution is $P^i/P^j = \alpha^i/a^j$ $(i,j=1,\ldots,I)$. Using (1) we obtain $Q^i = \beta^i$ $(i=1,\ldots,I)$.

MT3: The system of equations (44a, b) remains invariant. Thus the new and old purchasing power parities are equal to each other, and (51) follows immediately.

MT4: The transformation leaves all value shares $w_n^i$ invariant. The system of equations (44a, b) then implies that $P^i$ transforms into $\alpha^i P^i$ $(i=1,\ldots,I)$. Using (1) we obtain the desired result.

MT5 can be verified directly.

MT6 and MT7 are satisfied by definition.

With obvious modifications the proofs for MT8 and MT9 run parallel to those in the proof of Proposition 5. QED

## 6. References

Balk, B.M. (1989). On Van IJzeren's Approach to International Comparisons and Its Properties. Statistical Papers/Statistische Hefte, 30, 295–315.

Balk, B.M. (1996). Van IJzeren's Method of International Price and Volume Comparison: An Exposition. In International Comparisons of Prices, Output, and Productivity, edited by D.S.P. Rao and J. Salazar-Carrillo. Amsterdam: North-Holland.

Caves, D.W., Christensen, L.R., and Diewert, W.E. (1982). Multilateral Comparisons of Output, Input, and Productivity using Superlative Index Numbers. The Economic Journal, 92, 73–86.

Diewert, W.E. (1986). Microeconomic Approaches to the Theory of International Comparisons. Technical Working Paper No. 53. Cambridge MA: National Bureau of Economic Research. Abridged version, entitled "Test Approaches to International Comparisons," in Eichhorn (1988) and in Diewert and Nakamura (1993).
Diewert, W.E. (1987). Index Numbers. In The New Palgrave: A Dictionary of Economics, Volume 2, edited by J. Eatwell et al. New York: Macmillan. Reprinted in Diewert and Nakamura (1993).
Diewert, W.E. (1995). Price and Volume Measures in the System of National Accounts. Discussion Paper 95-02. Vancouver: Department of Economics, The University of British Columbia.
Diewert, W.E. and Nakamura, A., editors (1993). Essays in Index Number Theory, Volume 1. Amsterdam: North-Holland.
Dikhanov, Y. (1994). Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures. Draft Report. Washington DC: The World Bank. Presented at the 23rd General Conference of the International Association for Research in Income and Wealth, St. Andrews, Canada.
Drechsler, L. (1973). Weighting of Index Numbers in Multilateral International Comparisons. The Review of Income and Wealth, 19, 17–34.
Drechsler, L. and Krzeczkowska, E. (1982). Purchasing Power Parities in International Comparisons: Quantity vs. Price Comparisons. The Review of Income and Wealth, 28, 253–259.
Eichhorn, W., editor (1988). Measurement in Economics. Heidelberg: Physica-Verlag.
Eltetö, Ö. and Köves, P. (1964). On an Index Computation Problem in International Comparisons (in Hungarian). Statisztikai Szemle, 42, 507–518.
Eurostat (1977). Comparison in Real Values of the Aggregates of ESA 1975. Luxemburg: Statistical Office of the European Communities.
Eurostat (1983). Comparison in Real Values of the Aggregates of ESA 1980. Luxemburg: Statistical Office of the European Communities.
Fisher, I. (1922). The Making of Index Numbers. Boston: Houghton Mifflin.
Geary, R.C. (1958). A Note on Comparisons of Exchange Rates and Purchasing Power Between Countries. Journal of the Royal Statistical Society, Series A, 121, 97–99.
Gerardi, D. (1974). Sul Problema della Comparazione dei Potesi d'Acquisto della Valute. Padova: Istituto di Statistica dell' Universita di Padova.
Gini, C. (1924). Quelques Considérations au Sujet de la Construction des Nombres Indices des Prix et des Questions Analogues. Metron, 4, 3–162.
Green, J. and Heller, W.P. (1981). Mathematical Analysis and Convexity with Applications to Economics. In Handbook of Mathematical Economics, Volume 1, edited by K.J. Arrow and M.D. Intriligator. Amsterdam: North-Holland.
Hill, R.J. (1995). Purchasing Power Parity Methods of Making International Comparisons. Dissertation. Vancouver: University of British Columbia.
Iklé, D.M. (1972). A New Approach to the Index Number Problem. Quarterly Journal of Economics, 86, 188–211.
Khamis, S.H. (1970). Properties and Conditions for the Existence of a New Type of Index Numbers. Sankhyā, Series B, 32, 81–98.

Khamis, S.H. (1972). A New System of Index Numbers for National and International Purposes. Journal of the Royal Statistical Society, Series A, 135, 96–121.

Khamis, S.H. (1984). On Aggregation Methods for International Comparisons. The Review of Income and Wealth, 30, 195–205.

Khamis, S.H. and Rao, D.S.P. (1989). On a Gerardi Alternative for the Geary-Khamis Measurement of International Purchasing Powers and Real Product. Journal of Official Statistics, 5, 83–87.

Kloek, T. and Theil, H. (1964). International Comparisons of Prices and Quantities Consumed. Econometrica, 33, 535–556.

Köves, P. (1995). The EKS-Indices Applied for International Comparison (in Hungarian). Statisztikai Szemle, 73, 5–30.

Kravis, I.B., Heston, A., and Summers, R. (1978). International Comparisons of Real Product and Purchasing Power. Baltimore, MD: Johns Hopkins University Press.

Kravis, I.B., Heston, A., and Summers, R. (1982). World Product and Income: International Comparisons of Real Gross Product. Baltimore, MD: Johns Hopkins University Press.

Kravis, I.B., Kenessey, Z., Heston, A., and Summers, R. (1975). A System of International Comparisons of Gross Product and Purchasing Power. Baltimore, MD: Johns Hopkins University Press.

Kurabayashi, Y. and Sakuma, I. (1981). An Alternative Method of Multilateral Comparisons of Real Product Constrained with the Matrix Consistency. Paper presented at the 17th General Conference of the IARIW, Gouvieux, France.

Kurabayashi, Y. and Sakuma, I. (1982). Transitivity, Factor Reversal Test and Matrix Consistency in the International Comparisons of Real Product. Discussion Paper 54. Tokyo: Institute of Economic Research, Hitotsubashi University.

Kurabayashi, Y. and Sakuma, I. (1990). Studies in International Comparisons of Real Product and Prices. Tokyo/Oxford: Kinokuniya Company/Oxford University Press.

Nuxoll, D.A. (1994). Differences in Relative Prices and International Differences in Growth Rates. The American Economic Review, 84, 1423–1436.

Rao, D.S.P. (1971). On the Existence and Uniqueness of a New Class of Index Numbers. Sankhyā, Series B, 33, 341–354.

Rao, D.S.P. (1990). A System of Log-Change Index Numbers for Multilateral Comparisons. In Comparisons of Prices and Real Products in Latin America, edited by J. Salazar-Carrillo and D.S.P. Rao. Amsterdam: North-Holland.

Rao, D.S.P. and Banerjee, K.S. (1986). A Multilateral Index Number System Based on the Factorial Approach. Statistische Hefte/Statistical Papers, 27, 297–313.

Rao, D.S.P. and Salazar-Carrillo, J. (1988). A General Equilibrium Approach to the Construction of Multilateral Index Numbers. In World Comparison of Incomes, Prices and Product, edited by J. Salazar-Carrillo and D.S.P. Rao. Amsterdam: North-Holland.

Szulc, B. (1964). Index Numbers of Multilateral Regional Comparisons (in Polish). Przeglad Statysticzny, 3, 239–254.

Takayama, A. (1974). Mathematical Economics. Hinsdale: The Dryden Press.

Theil, H. (1960). Best Linear Index Numbers of Prices and Quantities. Econometrica, 28, 464–480.

United Nations (1986). World Comparisons of Purchasing Power and Real Product for 1980. New York: United Nations.

United Nations (1992). Handbook of the International Comparison Programme. Studies in Methods Series F No. 62. New York: United Nations.

van IJzeren, J. (1956). Three Methods of Comparing the Purchasing Power of Currencies. Statistical Studies No. 7. Zeist: The Netherlands Central Bureau of Statistics/De Haan.

van IJzeren, J. (1983). Index Numbers for Binary and Multilateral Comparison; Algebraical and Numerical Aspects. Statistical Studies No. 34. The Hague: The Netherlands Central Bureau of Statistics/Staatsuitgeverij.

van IJzeren, J. (1987). Bias in International Index Numbers: a Mathematical Elucidation. Private edition.

van IJzeren, J. (1988). Weighting and Additivity Problems of Multilateral Comparison. In Eichhorn (1988).

Received March 1994
Revised September 1995