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Computing Elementary Aggregates in the Swedish Consumer Price Index

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Abstract: This report presents the philosophy and approach taken by Statistics Sweden in revising the computation methods for elementary (lowest level) aggregates in its Consumer Price Index (CPI). A distinction is made between (a) the ideal definition, (b) the operational parameter and (c) the estimator of the elementary aggregate. Different alternatives for operational parameters are discussed with respect to their approximation properties and performance with respect to Fisher-type tests. One of these tests, the permutation test, is new. The parameter which was adopted in the Swedish CPI in 1990 is presented; this parameter has not, as far as we know, been used earlier.

Key words: Index tests; chain index; Edgeworth index.

1. Introduction

In 1989–1991 some significant changes took place in the Swedish Consumer Price Index (CPI). A new computation method for elementary aggregate indices was adopted, a new sampling and estimation system was launched – see Ohlsson (1990) – and a new method of measuring prices and calculating indices for clothing items was introduced for 1991.

This report deals with the first of these changes – the new index formula for elementary aggregates.

A CPI system consists of a hierarchy of price observations for items, combined into item groups at successively higher levels. At the highest level different goods and services are weighted together, usually according to the Laspeyres' or a fixed quantity index, using information from, e.g., household expenditure surveys and also possibly, as in Sweden, the National Accounts.

At lower levels most countries have to manage with less relevant weighting information or simply use unweighted measures. (The U.S. CPI seems to be an exception due to its very large budget.) For example, in Sweden there is no information on what items are sold in a particular outlet, let alone the quantities sold. This gives rise to the problem of computing *elementary* (or basic) *aggregates*. An elementary aggregate is defined as the first level at which price observations are combined. Above, but not below, this level we assume that there is information on quantities or values that can be used in the usual index formulas. Within

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an elementary aggregate we have only crude weights (such as the size of an outlet in terms of turnover or number of employees) or no weights at all.

In the last ten years or so a number of papers on this topic have emerged, see, e.g., Forsyth (1978), Carruthers, Sellwood and Ward (1980), Morgan (1981), Szulc (1983, 1989), and Turvey et al. (1989).

In this paper we present the philosophy and approach to these problems that has been adopted in the Swedish CPI. We believe that much of this philosophy might also be of interest for other countries although circumstances are likely to differ with respect to the details in aggregation procedures, available data, etc.

After explaining notation in Section 2 we give a brief introduction to the basic structure of the Swedish CPI in Section 3. In Section 4 we present an approach to the conceptual problems that we think are involved in the choice of elementary aggregate formula. In Section 5 we discuss the definition and in Section 6 the path from a definition to an operational parameter for an elementary aggregate. In Section 7 index tests for elementary aggregates are formulated and Section 8 presents the various index formulas that have been considered. In Section 9 some further aspects of the parameter problem, relating to the two-dimensional nature of the population, are discussed. In Section 10 some mathematical relations between the parameters are given (most proofs are deferred to an Appendix). Finally, in Section 11, data and specific circumstances in the Swedish CPI are further discussed and in Section 12 conclusions are drawn.

2. Notation

We use upper case letters P, Q, and W to denote prices, quantities, and weights at an aggregate level. Lower case letters p, q, and w are used at the basic level denoting prices, etc., for a variety in a single outlet.

Subscripts denote time periods expressed as year (,month). For example, $P_{t,m}$ means the average price in month m of year t while as Q_{t-1} means aggregate quantity in the whole year $t-1$. An index number, denoted I is characterized by a subscript denoting reference period and a superscript denoting comparison period. Later, from Section 5 and onwards, where we are only interested in elementary aggregates we use a simplified form of index notation with I_{0t} meaning an index from period 0 to period t.

Subscripts for goods and services are generally suppressed.

Index numbers will always be defined without the normation to 100. That is, the value 1 means no change.

3. Basic Structure of the Swedish CPI

The Swedish CPI is a chain index with annual links. This means that the annual links are multiplied together when calculating indexes for longer periods. Every link compares prices in the base month of December to those in a certain month in the following year. For each new link, weights are recalculated based on new information.

We make a distinction between a long-term link (L), which in principle uses quantity weights Q_t from year t and a short-term link (S) which uses quantity weights Q_{t-1} from year $t-1$. The definitions of the links are

$$L_{t-1,\mathrm{Dec}}^{t,\mathrm{Dec}} = \frac{\sum P_{t,\mathrm{Dec}} Q_t}{\sum P_{t-1,\mathrm{Dec}} Q_t} \quad (1)$$

and

$$S_{t-1,\mathrm{Dec}}^{t,m} = \frac{\sum P_{t,m} Q_{t-1}}{\sum P_{t-1,\mathrm{Dec}} Q_{t-1}} \quad (2)$$

where as usual $P_{t,m}$ is the price for month m of year t and Σ stands for summation "over all item groups" (see Section 5 below) bought by private consumers.

The chained index from December year 0 to month t, m then becomes

$$I^{t,m}_{0,\mathrm{Dec}} = S^{t,m}_{t-1,\mathrm{Dec}} \prod_{j=1}^{t-1} L^{j,\mathrm{Dec}}_{j-1,\mathrm{Dec}}. \qquad (3)$$

In order to give the whole chain a base year (at present we have 1980 = 100) an additional factor measuring price changes from the average of the base year up to December is multiplied.

Note that, in the long run, the Swedish CPI is only dependent on the long-term links, since the short-term links are successively replaced by their long-term counterparts. This is done in the beginning of each year. For this reason we actually have two CPIs for December, one based on the short-term link and one on the long-term link.

This system – used in Sweden since the middle of the 1950s – is a way of avoiding the well-known Laspeyres' upward and Paasche's downward bias since the weights represent a whole year which essentially lies *between* the reference and the comparison period. The Swedish CPI is therefore close to a chained Edgeworth index.

The price paid for this formula advantage is that weights have to be estimated based on data that are perhaps weaker than in other countries. They are calculated mainly from consumption values provided by the National Accounts and divided into about 90 categories. For more detailed item groups and for housing, other sources are used for the weighting, which are not updated every year. The National Accounts make preliminary estimates for the first three quarters of a year and then add projections for the fourth quarter.

The actual computation of the aggregate indices is more complicated as will be explained here since the weights provided by the National Accounts must be recalcualted to represent the correct period. The procedure is as follows:

Both S and L could be written as

$$\sum W \cdot I^{t,m}_{t-1,\mathrm{Dec}}$$

with

$$W = \frac{V_b \dfrac{P_{t-1,\mathrm{Dec}}}{P_b}}{\sum V_b \dfrac{P_{t-1,\mathrm{Dec}}}{P_b}}, \quad I^{t,m}_{t-1,\mathrm{Dec}} = \frac{P_{t,m}}{P_{t-1,\mathrm{Dec}}} \qquad (4)$$

and b standing for year $t-1$ in S and t in L. In L month m always stands for December. Here $V_b = P_b Q_b$ are the weights from the National Accounts and other sources.

Now, in S where $b = t-1$ we have

$$\frac{P_{t-1,\mathrm{Dec}}}{P_{t-1}} = \frac{\dfrac{P_{t-1,\mathrm{Dec}}}{P_{t-2,\mathrm{Dec}}}}{\dfrac{P_{t-1}}{P_{t-2,\mathrm{Dec}}}} \approx \frac{S^{t-1,\mathrm{Dec}}_{t-2,\mathrm{Dec}}}{\dfrac{1}{12}\sum_{m=1}^{12} S^{t-1,m}_{t-2,\mathrm{Dec}}} \qquad (4a)$$

and in L where $b = t$ we have

$$\frac{P_{t-1,\mathrm{Dec}}}{P_t} \approx \frac{1}{\dfrac{1}{12}\sum_{m=1}^{12} S^{t,m}_{t-1,\mathrm{Dec}}}. \qquad (4b)$$

(4), (4a) and (4b) specify the actual computation of the weights, W, completely.

The item group indices $I^{t,m}_{t-1,\mathrm{Dec}}$, on the other hand, are defined and computed in different ways, depending on the type of item and on the price and weight information available in each case. In this paper we primarily discuss items in the LOCAL PRICE SYSTEM (LOPS) and the LIST PRICE SYSTEM (LIPS), where the elementary aggregate problem is most important. These systems together account for about 46% of the total weight of the Swedish CPI.

In LOPS, which covers clothing items, fresh fruit and vegetables, furniture, household appliances, etc., prices are collected by field interviewers each month. There is a stratification of outlets by the classification in the business register and of items into item groups which are fairly narrow. As weights for outlet strata within item group we use cross-classified data from surveys made every third to fifth year. As outlet weights within an outlet stratum, the number of employees is used. The sampling of items and varieties is purposive because of the lack of a sampling frame, while as the sampling of outlets is random with probabilities proportional to size.

In LIPS, which covers most food items and other daily necessities, prices are taken from the regional price lists (except for December each year, when actual outlet prices are collected) of three chains of wholesalers in Sweden. Items are stratified into item groups. There are historic data about sales values of all varieties which are used for employing systematic sampling of varieties with probabilities proportional to the sales values of the varieties. Outlets are stratified regionally and within an outlet stratum sampling is with probabilities proportional to size.

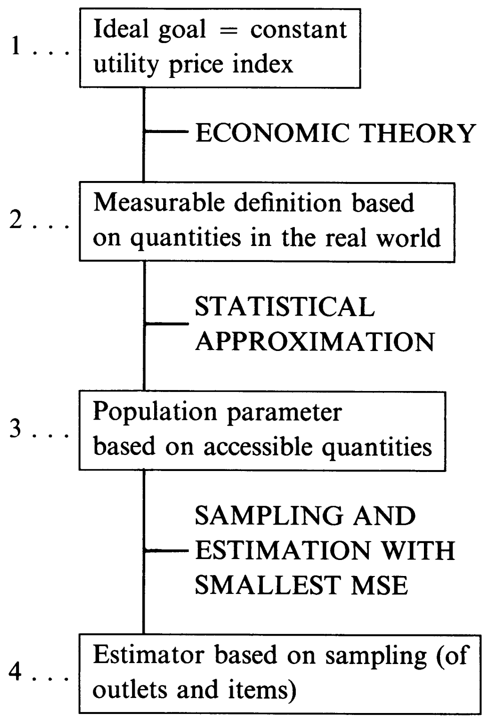
The elementary aggregate problem both in LOPS and in LIPS is how to aggregate price observations for items and varieties within an item group and outlet stratum. It is not possible simply to proceed from the general definition in (1) or (2) since quantity information is lacking and we do not have access to weight information for the correct weight base period.

4. An Approach to the Conceptual Problems in the CPI

In a statistical survey in general we can formulate the trilogy: Ideal Goal – Statistical Goal – Estimate from a Sample.

Here the ideal goal relates to the underlying problem of the users of the survey. The statistical goal is the ideal goal translated into the statistical world of populations and measurable quantities and is a property defined for an entire target population. Finally, from this population, a sample is drawn and an estimate is made.

For a CPI the following scheme, which involves four levels would provide an analogy:



The first level in this scheme is somewhat controversial for several reasons. Firstly, there is not, even theoretically, any such thing as a constant utility price index (or, as it is also called, a cost-of-living index) defined for a group of people, let alone the whole population of a country. Allen (1975) calls it "an act of faith" to go from the constant utility price index for an individual to an aggregate, social price index. Later theoretical attempts in this direction have been made, however, for example by Jorgenson and Slesnick (1983) and by Blackorby and Donaldson (1983). One issue here is whether the index should be democratic or plutocratic, which refers to whether individuals should be equi-weighted or weighted on the basis of their consumption. Practical index construction approximates a plutocratic index.

Secondly, there seems to be different opinions as to the value of basing official price index calculations on this theoretical concept at all. For example, in ILO (1987) – a resolution concerning consumer price indices adopted by the Fourteenth International Conference of Labour Statisticians – no reference to the cost-of-living concept is made. Instead, there is the following statement of intent (page 124):

> "The purpose of a consumer price index is to measure changes over time in the general level of prices of goods and services that a reference population acquire, use or pay for consumption."

On the other hand BLS (1988), the official handbook of methods for the U.S. CPI, explicitly refers to the concept of the cost-of-living index as

> "a unifying framework for dealing with practical questions that arise in construction of the CPI."

We basically agree with this latter point of view since it is necessary to have a theoretical point of departure in situations of quality change and changes in the sets of items and outlets that we measure. However, references to the cost-of-living index have to be mainly intuitive.

The second level in the scheme refers to a concept in which all components are in principle observable in the physical world (unlike the constant utility price index, where utility is unobservable). This definition should be in terms of detailed data (prices and quantities) on all transactions taking place in the market and must go all the way from a price observation for an item in an outlet up to the all items CPI. Essentially this is a task for economic research, where more effort should be expended.

The third level, the population parameter, refers to a concept based on those quantities that are accessible in the statistical practice with budget restrictions, etc. These quantities are considerably less detailed and in some respects not quite up-to-date. Typically we do not have relevant quantity measures to attach to single price observations but we may have more crude size measures like total sales of an outlet or an item. This statistical parameter should ideally be arrived at through a series of approximations from a measurable definition in step 2 or, perhaps, directly from the basic definition in step 1. Since, as we will see, such an approximation philosophy does not lead to a definite answer there is also a need for an alternative approach, the so called axiomatic approach, based on index tests.

The fourth level is the result of a traditional statistical sampling procedure. Probability sampling is of course the preferred technique but sometimes frame problems and the necessity of obtaining comparable price observations at two time periods make purposive, balanced samples the only possibility, particularly in the item dimension. Based on some cost restriction, the variance or the mean square error of the estimator (in relation to the population parameter) should be minimized.

In summary we look at the constant utility price index as the Ideal Goal of the CPI in spite of the difficulties in defining this concept for a group of people. It is not, however, possible to clearly point at one well-defined Statistical Goal. Instead, this goal will differ between different goods and services for reasons that have to do with the type of consumption involved as well as with the kind of data available.

5. The Definition of a CPI

Here and in the following sections we consider index numbers from a reference period 0 to a comparison period t of an arbitrary year, i.e., links of a chain index. In the Swedish CPI 0 will be December and t a month of the following year. Note the change in notation from here on.

5.1. The basic definition

The basic definition of almost any CPI (or, to be exact, of an index link) such as (1) and (2) above is of the fixed-quantity type

$$\mathrm{CPI}_{0t} = \frac{\sum P_t Q_b}{\sum P_0 Q_b} = \sum W I_{0t} \qquad (5)$$

where summation is over item groups, b is the expenditure base period, 0 is the reference period and t is the comparison period. $W = P_0 Q_b / \Sigma P_0 Q_b$ and $I_{0t} = P_t / P_0$. At some points in time (in Sweden annually) the expenditure base is updated and indices are chained. If $b \leqslant 0$, this definition is generally believed to be an over-estimate of a constant utility price index due to the consumer's ability to shift from a higher priced good to a lower priced good that yields the same utility.

This is really only a beginning of a CPI definition and leaves many problems unsolved, for example:

1. It presupposes an all-inclusive item grouping related to the expenditure base period. It does not tell us what to do below that level, for which we have relevant weights.
2. It does not deal with the problem of quality change, that is, what to do when we no longer have an item identical to that in period b. Different practices have evolved here, like different variants of linking or imputation, direct quality evaluation or hedonic regression. None of them could be motivated by definition (5) but only by reference to the underlying concept of the cost-of-living index.
3. It does not deal with the problem of new or disappearing items. There is a need for some theoretical guidance, for example, as to when an item should be considered a new product as opposed to a new variant of an old product with, perhaps, a new quality. This question is, in a sense, impossible for economic theory proper, since the cost-of-living index is generally not even separable into different indices for different goods and services.
4. It does not deal with the problem of new or disappearing outlets. If, for example, a new discount store opens (or an old one closes), its prices are, in standard CPI practice, not compared with those of the other outlets with the result that its direct effect on the price level is not estimated. Theoretically there could not be a good motive for this practice, even though one may consider the new store to have a different quality of service from the old ones. The risk for systematic errors is still considerable with the present practice.
5. It does not tell us how to handle items for which the price changes within the reference/comparison period in the same outlet. Here there is usually a purposive sampling in time of one day in the period, in Sweden a day in the middle week of the month. But theoretically, is it not the monthly average price that we are supposed to measure?

In all these five respects there is neither a theoretical nor a practical consensus of what to do when calculating consumer price indices.

5.2. The definition of an elementary aggregate

We could look at equation (5) and its I_{0t} in two ways, dividing the basic definition into two variants.

Definition 5A: We think of the summation in (5) as applying only to some basic division of private consumption into elementary aggregates (which may be item groups like milk, newspapers, haircuts, etc.). For one such aggregate we interpret the prices – P_t and P_0 – as mean prices. These are defined as quantity weighted arithmetic means of all prices paid for all items within the group, in all outlets and during all days in the period (usually a month). This is actually the definition of a unit value index

$$I_{0t} = \frac{\sum' q_t p_t / \sum' q_t}{\sum'' q_0 p_0 / \sum'' q_0}. \qquad (5a)$$

(Note that in (5a) the summations in the numerator and the denominator are over *different sets* of items and outlets, due to the changes in item and outlet structure between 0 and t). Formula (5a) should be the best definition for homogeneous item groups. But for non-homogeneous groups it is vulnerable to quality differences. In order to estimate it in a straight-forward manner a sampling philosophy different from the one usually applied would be required. We would, e.g., have to have separate item and outlet samples for each time period.

Definition 5B: We think of the summation as being over all item/outlet pairs in period b. This is, for example, the viewpoint of Gillingham (1974). The P and Q are then interpreted to be quantities at the micro level, giving

$$I_{0t} = \frac{\sum q_b p_t}{\sum q_b p_0}. \qquad (5b)$$

(Note that in (5b) the summation is over *the same set* of items and outlets as in the numerator.)

This is a very consistent definition but it has several shortcomings which are difficult and expensive to resolve, at least for small countries with a limited CPI budget. First, it does not take care of varying prices for the same item in the same outlet during the reference period – you would still have to define a mean price within that period in some way (or consider 0 and t to be fixed points in time rather than periods which would give you a less relevant index). Second, it is not possible to apply 5B in practice unless b is sufficiently far before 0. Third, given that b is far before 0, 5B accentuates the tendency of a fixed quantity index to overestimate a constant utility price index since it does not take into consideration the possibility that a consumer might switch outlets to take advantage of reduced prices. Fourth, the problem of linking (or not linking) new items and outlets to old ones remains.

In Sweden (and presumably most other countries as well, the U.S.A. probably being a major exception) neither of these definitions could be rigorously applied in practice without taking more approximation steps. It is too expensive to collect data for constructing an item/outlet matrix for the whole population with information on which outlets sell what items in what quantities. This could be done only for sampled items and outlets. An interesting future perspective, however, is the introduction of computerized cashier counters in many stores. If traded quantities are registered this way, it would enable us to work with more sophisticated parameters in our price indices!

6. From Definition to Parameter

For reasons discussed above we have to base our index calculations for elementary aggregates on a simple averaging formula. Therefore we have to find a rationale for our choice of this formula.

As in any survey, we make a distinction between the population parameter and the estimator. The population parameter is considered to be the quantity to be estimated, or what we would have obtained if we had been able to measure all outlets (and in LIPS also all items) in the population. In the case of price indices the parameter is equivalent to the basic index formula. The estimator is then determined using sampling theory and depends on the sampling design used (in our case PPS).

For elementary aggregates we typically do not have any useful quantity weights. Instead we have crude size measures related to the total sales of an outlet or no weights at all. (Some methodologists – e.g., Allen (1975) – refer to indices without quantity weights as "stochastic.") In the Swedish CPI we use size measures which are related to total sales (i.e., price*quantity) of an outlet or an item in a certain time period.

Let us define a price index in this context as a function, I, of two price vectors and one weight vector.

$$I = I(\mathbf{w}, \mathbf{p}_t, \mathbf{p}_0) \tag{6}$$

where $\mathbf{w} = (w_1, \ldots, w_N), \mathbf{p}_t = (p_{t1}, \ldots, p_{tN})$ and $\mathbf{p}_0 = (p_{01}, \ldots, p_{0N})$. Now, our first choice for finding a reasonable parameter is trying to approximate a definition like 5A or 5B in Section 5.

Starting from definition 5A the index could be expressed as a ratio of mean prices weighted by the quantity q bought at each price. This makes it possible to pursue the following series of approximations. The first of these is the fact that an average price is, in the absence of the relevant quantity weights, best computed as a harmonic mean. This point is further explained in Ohlsson (1989) and the argument is

$$\bar{p} = \frac{\sum qp}{\sum q} = [w = pq] = \frac{\sum w}{\sum w/p}. \tag{7a}$$

This holds exactly if the w are the actual sales values. This is not often the case in practice but if the w are correlated with sales (7a) is likely to be a better approximation than, e.g., the weighted arithmetic mean.

Now, using (7a) we obtain

$$\frac{P_t}{P_0} = \frac{\bar{p}_t}{\bar{p}_0} \approx \frac{\sum' w_t / \sum' w_t/p_t}{\sum'' w_0 / \sum'' w_0/p_0} = \frac{\sum'' \frac{w_0}{\sum'' w_0} \Big/ p_0}{\sum' \frac{w_t}{\sum' w_t} \Big/ p_t} = H_A$$

$$\approx \left[\begin{array}{c}\text{if the population of outlets}\\ \text{is unchanged and the weight}\\ \text{structures are similar}\end{array}\right]$$

$$\approx \frac{\sum w/p_0}{\sum w/p_t} = H. \tag{7b}$$

Here we have two levels of approximation – H_A and H. The H_A is closer to definition 5A and applying it means having different sets of weights and outlets in the numerator and the denominator. If there is unit price elasticity of demand then H_A results exactly. In H, we further simplify by using the same weights and outlets. The result of this is that we do not capture the effects of changes in the set of outlets or items. Szulc (1989) notes that the equiweighted harmonic mean is an alternative for homogeneous items with frequent price wars.

On the other hand, starting from definition 5B above the fixed weight principle is considered applicable down to the very lowest level of an item/outlet pair. In this case we could use the following series of approximations

$$\frac{\sum q_b p_t}{\sum q_b p_0} = \frac{\sum q_b p_b (p_t/p_b)}{\sum q_b p_b (p_0/p_b)}$$

$$\approx [w_b \approx p_b q_b] \approx \frac{\sum w_b (p_t/p_b)}{\sum w_b (p_0/p_b)}. \quad (8)$$

Now, in the Swedish long-term link, p_b is the average price over the period from 0 to t. It can be approximated by, e.g., $p_b = (p_0 + p_t)/2$ (another, slightly better but less practical, approximation would be $p_b = (p_{\text{Jan}} + p_{\text{Feb}} + \ldots + p_{\text{Dec}})/12$) leading to

$$RA = \frac{\sum w_b p_t/(p_0 + p_t)}{\sum w_b p_0/(p_0 + p_t)}. \quad (8a)$$

Using definition 5B we thus arrive at another operational parameter, called RA, after a series of suitable approximations. This approximation is especially geared to the type of base period used in the definition of the Swedish long-term link.

7. Index Tests

Since the times of Irving Fisher (1922) one strand of index theory – the so-called axiomatic approach – has been to formulate certain tests (or axioms) that the various index formulae should pass. A later paper in this tradition is Eichhorn and Voeller (1983). They define a price index to be a mathematical function of 2 × 2 vectors of prices and quantities at the two time-periods with N elements each. This function is to fulfill certain tests of a completely mathematical nature.

For elementary aggregates there is one fundamental difference compared with the Eichhorn and Voeller system: no quantities are available. We formulate the following six tests, considering them to be the most essential ones and relate their properties to the Eichhorn and Voeller system.

I. Monotonicity Test

The function I is strictly increasing with respect to $\boldsymbol{p}_t$ and decreasing with respect to $\boldsymbol{p}_0$:

$$I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0) > I(\boldsymbol{w}, \boldsymbol{p}'_t, \boldsymbol{p}_0) \text{ if } \boldsymbol{p}_t \geqslant \boldsymbol{p}'_t$$

and

$$I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0) < I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}'_0) \text{ if } \boldsymbol{p}_0 \geqslant \boldsymbol{p}'_0. \quad (9)$$

Interpretation: If at least one price ratio increases (decreases) while the others remain equal then the price index also increases (decreases). (The sign $\geqslant$ means that all arguments are larger and at least one argument is strictly larger.)

II. Proportionality Test

If all corresponding prices differ by the same factor c, then the value of I equals c:

$$I(\boldsymbol{w}, c\boldsymbol{p}_0, \boldsymbol{p}_0) = c. \quad (10)$$

III. Price Dimensionality Test

The same proportional change in the unit of the currency does not change the value of I:

$$I(\boldsymbol{w}, c\boldsymbol{p}_t, c\boldsymbol{p}_0) = I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0). \quad (11)$$

This test may appear to do the same thing as the Proportionality Test. However, Eichhorn and Voeller (1983, p. 420) give an example of a function passing II but not III, which also applies to our elementary aggregate situation. Test III is, however, a special case of IV below.

IV. Change-of-Units Test

The same change in the units of measurement of the corresponding items does not change the value of I.

$$I(\boldsymbol{w}, c_1 p_{t1}, \ldots, c_N p_{tN}, c_1 p_{01}, \ldots, c_N p_{0N}) = I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0). \quad (12)$$

This test gets a different formulation in the absence of quantity weights. The test is fulfilled by all functions that only use the price ratios and not the prices themselves.

V. (Price/Time) Reversal Test

Eichhorn and Voeller (1983) formulate two reversal tests: the Time Reversal and the Price Reversal Tests. These two tests coincide in the absence of quantity weights. We get the following formulation of the Reversal Test:

$$P(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0)P(\boldsymbol{w}, \boldsymbol{p}_0, \boldsymbol{p}_t) = 1. \quad (13)$$

This means that the price index calculated backwards from t to 0 with the same weights as the forwards index should be the inverse of the forwards index.

VI. Permutation (Price Bouncing) Test

If all $w_j = 1/N$ and if $\boldsymbol{p}'_t$ and $\boldsymbol{p}'_0$ are arbitrary permutations of $\boldsymbol{p}_t$ and $\boldsymbol{p}_0$ respectively, then $I(\boldsymbol{w}, \boldsymbol{p}'_t, \boldsymbol{p}'_0) = I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0)$. (14)

This very strong test has, as far as we know, not been proposed earlier. The practical need of a test like this stems from the "price bouncing" behaviour of the outlets. Price bouncing is a frequent phenomenon in Sweden and probably also in many other countries – see, e.g., Szulc (1989, p. 175). It occurs when prices for an item move up and down from month to month due to seasonal variation, bargain prices or substitutions. The movements could be tens and in extreme cases even hundreds of percent.

A special case of this test is the following:

VI-B: If all $w_j = 1/N$ and if $\boldsymbol{p}_t$ is a permutation of $\boldsymbol{p}_0$ then $I(\boldsymbol{w}, \boldsymbol{p}_t, \boldsymbol{p}_0) = 1$. (15)

This is a special case if also the proportionality test is fulfilled with $c = 1$. Perhaps this special case is enough for practical purposes.

What VI-B says is that the price index should show no change if prices "bounce" in such a manner that the outlets are just exchanging prices with each other. This could, for example, be the result of outlets moving up and down from ordinary to bargain prices so that there is the same set of prices in the market in both time periods but in different outlets – a common practice in Sweden. If all weights are equal, it is intuitively obvious that the price level has not changed in this case.

8. Operational Parameters for Elementary Aggregates

In this section we assume that there is only one summation level – over outlets for a specific item like in the Swedish LOPS. The parameters are easily generalized to cases where the outlets are stratified or clustered. The case of summation also over items in an item group is discussed in the next section.

The weights are normed so that $\Sigma w = 1$. For those parameters where we have more than one summation, all summations are over the same set of outlets.

8.1. The ratio of mean prices

This parameter (labelled A) has the following form

$$A_{0t} = \frac{\sum wp_t}{\sum wp_0}. \quad (16)$$

It is simple to understand and seems to be the most popular elementary aggregate formula on a global scale. It was used in the Swedish CPI up to 1989.

Still it has one considerable shortcoming. If the weights w are related to sales amounts, then $w \approx p_b q_b$, where b is a weight base period and

$$A_{0t} \approx \frac{\sum q_b p_b p_t}{\sum q_b p_b p_0}. \quad (17)$$

This means that – compared with the basic definition (5) there is a weighting with p_b in addition to the quantity weight. This is absurd and means that in effect A overweights outlets with higher prices. This leads to an index relevant for luxury consumers!

From the point of view of the above index tests, A fulfills all of them except the Change-of-Units Test. In practice this means that, if, for example, one outlet sells eggs per dozen instead of per kg and the price is rescaled to the other unit, this would change the resulting index number if A is used.

The shortcomings of A must be judged to be serious. It must be noted, however, that A is the natural choice if the w were quantity weights. If these weights also represent the weight base period b, the A parameter corresponds to the basic CPI definition (5), which meets the Change-of-Units Test.

8.2. *The mean of price ratios (Sauerbeck index)*

This parameter (labelled R) has the following form

$$R_{0t} = \sum w \frac{p_t}{p_0}. \tag{18}$$

The parameter R is also simple to understand and widely used. It was, unfortunately, used in the Swedish CPI (short-term link) from January to March 1990. It does not have the problem of the A parameter – low and high prices influence the index to the same extent and consequently it meets the Change-of-Units Test. Still it was abandoned from April 1990 and onwards. Why?

The parameter R passes all of the above index tests except the Reversal Test and the Permutation Test. It is an easy mathematical exercise (see Section 10) to show that

$$R_{0t} \times R_{t0} \geqslant 1. \tag{19}$$

It is likewise possible (see 10.1.2 and Appendix) to show that if the prices at time t are a permutation of those at time 0 then

$$R_{0t} \geqslant 1. \tag{20}$$

In both (19) and (20) equality applies only if all prices are equal at 0 and t. These two relations show that R has an upward bias, which could be quite large in practice (see Section 11) due to price bouncing.

A simple example illustrates the effect. Say that the regular price of coffee is 25 SEK/$\frac{1}{2}$ kg. But coffee is a good that is often subject to a bargain price of, say, 20 SEK/$\frac{1}{2}$ kg. Now suppose that one store reduces its price from period 0 to period t in this way. This means a price ratio of 20/25 = 0.8. Another store with the same weight goes the other way and raises its price which gives a price ratio of 25/20 = 1.25. Averaging these price ratios gives (0.8 + 1.25)/2 = 1.025 – a price increase by 2.5%! Since the price level is unchanged this is absurd.

8.3. *The geometric mean*

This parameter (labelled G) has the following form

$$G_{0t} = \prod (p_t/p_0)^w = \frac{\prod (p_t)^w}{\prod (p_0)^w}. \tag{21}$$

Looking at these alternative expressions, one of its advantages is immediately seen – by taking geometric means you have an equivalence between means of ratios and ratios of means.

The parameter G also meets all of the above index tests – the Change-of-Units Test (which A fails) as well as the Reversal Test and the Permutation Test (which R fails).

It is an open question to what extent G could be motivated by economic theory. Pollak (1983) formulates conditions under which the cost-of-living index (COL) is a geometric mean. This depends on the form of the utility function and whether the COL is a function of price ratios only (and not prices in some other combination). In CPI practice, however, we take the basic definition (5) for granted and the issue of geometric means only applies to the elementary aggregate level. It is not possible to find a simple approximation method leading from (5) to G.

In a resolution of the Fourteenth International Conference of Labour Statisticians (ILO 1987) there is, however, a specific recommendation:

> "In the calculation of elementary aggregate indices, consideration should be given to the possible use of geometric means."

8.4. The ratio of harmonic means

This parameter (labelled H) has the following form

$$H_{0t} = \frac{\sum w/p_0}{\sum w/p_t}. \tag{22}$$

As already described in Section 6 this parameter is arrived at by approximating definition 5A above and could thus be an alternative for homogeneous item groups.

However, alike A, H does not meet the Change-of-Units Test. But contrary to A, H actually gives larger weight to outlets with lower prices. This could be seen by rewriting it as

$$H_{0t} = \sum (w/p_t)(p_t/p_0)\Big/\sum (w/p_t). \tag{23}$$

It thus becomes some kind of poor man's index, which may seem more attractive than the contrary. Still we consider this to be a shortcoming, since the w reflect all weight information that we want to use.

8.5. The harmonic mean of ratios

This parameter has the following formula

$$H'_{0t} = \frac{1}{\sum wp_0/p_t}. \tag{24}$$

It has the inverse properties of the R parameter. It passes all tests except for the Reversal Test and the Permutation Test. By observing that $H'_{0t} = 1/R_{t0}$ we immediately obtain that

$$H'_{0t}H'_{t0} < 1 \tag{25}$$

which means that it has a negative bias and underestimates a true index. It is therefore of no interest in itself.

8.6. The geometric mean of R and H′

The inverse properties of R and H' leads one to consider a combination of them. The natural combination is their geometric mean with the following formula

$$RH_{0t} = \frac{\left(\sum wp_t/p_0\right)^{1/2}}{\left(\sum wp_0/p_t\right)^{1/2}}. \tag{26}$$

RH passes all the above index tests except for the Permutation Test. It could be viewed as an approximation to Fisher's Ideal Index, which would result exactly if the w were value weights: in the numerator for period 0 and in the denominator for period t.

8.7. The ratio of normed mean prices

The last parameter to be discussed here is, as far as we know, a novel invention. We call it RA and its formula is

$$RA_{0t} = \frac{\sum wp_t/p_.}{\sum wp_0/p_.}, \text{ where } p_. = \tfrac{1}{2}(p_0 + p_t). \tag{27}$$

As described in Section 6 above this parameter is arrived at by approximating definition 5B for the long-term link.

The parameter RA passes all the above index tests except the Permutation Test. But unlike R it can be demonstrated that it has no definite upward or downward bias; RA could be larger as well as smaller than or equal to 1 in a permutation situation and differences are always small. We may therefore, somewhat loosely, state that RA (as well as RH) approximately passes the Permutation Test.

A disadvantage of RA is that the ratio of two RA parameters, e.g., one from 0 to t to one from 0 to t-1, is no longer an RA parameter in itself and does not pass the proportionality test. This ratio is used in the Swedish CPI to show the change in price level from t-1 to t. Numerical studies indicate that the deviations in this respect are negligible, however.

9. The Two-Dimensional Weight Structure

In the parameter formulations in Section 8, we have assumed that we have just one type of weight for every price observation. But reality is more complicated than that!

For many item groups there could be several weight/summation levels. However, two dimensions are basic: the item dimension and the outlet dimension.

So, now we consider a two-dimensional division of a certain item group according to consumption in the weight base period. In the vertical dimension we divide the item group into (homogeneous) items and in the horizontal dimension we have all outlets selling (in principle at least one of) the items in the item group. Ideally we now would like to have the complete two-dimensional weight structure, showing how much outlet j sold of item k. In practice many of these weights would be zero. This information is, at least in Sweden, not available. The weights that are available are marginal weights giving more or less good approximations to the relative sales of item k *or* of outlet j.

In addition to that, we are able to observe the zero weights *in the sample*, i.e., the outlets which do not trade item k (see 1_{kj} below). We have to take account of this additional information since an outlet with no price for item k automatically gets a zero weight for that item.

Let us use the following symbols: w_k is the weight of item k, v_j is the weight of outlet j, and 1_{kj} (an indicator) is 1 if item k is sold in outlet j and 0 otherwise.

(Theoretically the weights should be related to the base period b. The indicators should be 1 only if the item is sold in all three time periods b, 0, and t. But in practice the weights will refer to different time periods and there will be item substitutions which further obscure the matter.)

Because of this two-dimensional nature of the population, each parameter now divides into three "subparameters."

Let us take RA as an example. If we had the full weight matrix $\{w_{kj}\}$ we would have

$$RA = \frac{\sum_k\sum_j w_{kj}p_{1kj}/p_{\cdot kj}}{\sum_k\sum_j w_{kj}p_{0kj}/p_{\cdot kj}}. \qquad (28)$$

When we approximate with marginal weights we have three choices

$$RA_1 = \frac{\sum_k w_k\left\{\left(\sum_j v_j 1_{kj} p_{tkj}/p_{\cdot kj}\right)\Big/\left(\sum_j v_j 1_{kj}\right)\right\}}{\sum_k w_k\left\{\left(\sum_j v_j 1_{kj} p_{0kj}/p_{\cdot kj}\right)\Big/\left(\sum_j v_j 1_{kj}\right)\right\}} \qquad (29)$$

$$RA_2 = \frac{\sum_j v_j\left\{\left(\sum_k w_k 1_{kj} p_{tkj}/p_{\cdot kj}\right)\Big/\left(\sum_k w_k 1_{kj}\right)\right\}}{\sum_j v_j\left\{\left(\sum_k w_k 1_{kj} p_{0kj}/p_{\cdot kj}\right)\Big/\left(\sum_k w_k 1_{kj}\right)\right\}} \qquad (30)$$

$$RA_3 = \frac{\left(\sum_k\sum_j w_k v_j 1_{kj} p_{tkj}/p_{\cdot kj}\right)\Big/\left(\sum_k\sum_j w_k v_j 1_{kj}\right)}{\left(\sum_k\sum_j w_k v_j 1_{kj} p_{0kj}/p_{\cdot kj}\right)\Big/\left(\sum_k\sum_j w_k v_j 1_{kj}\right)}. \qquad (31)$$

(Note that in RA_3 but *not* in RA_1 or RA_2 a reduction of the fraction is possible.)

The difference between these alternatives is illustrated in the following example with three items and three outlets. The w_k are 0.4, 0.4 and 0.2 and the v_j are 0.6, 0.3 and 0.1, respectively, and cells with $1_{kj} = 0$ are marked with "–". Here we show the three weight structures that follow implicitly from using the above three alternatives

RA_1

			Sum
.24	.12	.04	.40
.40	–	–	.40
.13	.07	–	.20
.77	.19	.04	1

RA_2

			Sum
.24	.20	.10	.54
.24	–	–	.24
.12	.10	–	.22
.60	.30	.10	1

RA_3

			Sum
.29	.15	.05	.49
.29	–	–	.29
.15	.07	–	.22
.73	.22	.05	1

We observe that in RA_1 the item weights are preserved and the same is true for the outlet weights in RA_2. In RA_3, however, the weights are not preserved in any dimension.

A fourth alternative would be to work out an algorithm that preserves the weights in both dimensions and comes to a unique solution by minimizing some loss function. This alternative is, however, not feasible given the resources allocated to the Swedish CPI.

It is not easy to make a strong argument for any of these alternatives. One reason for choosing a certain alternative could be that certain weights are more important to preserve. For example, stratification (in the Swedish LIPS we stratify outlets by region) may call for preserving the weights in the stratification dimension.

This two-dimensional nature of the population also gives rise to important sampling, estimation and allocation problems, which are now examined at Statistics Sweden. Some contributions are given in Vos (1964). The author could provide more information about this work.

10. Mathematical and Statistical Properties of the Parameters

There are a number of important relations between the parameters above. From a mathematical point of view we note that the R, G, H', RH, and RA are all functions of the price ratios (p_t/p_0) alone, while A and H depend also on the price levels p_t and p_0 separately.

10.1. Inequalities

10.1.1. $R_{0t} \times R_{t0} \geqslant 1$ with equality if and only if all price ratios are equal.
Proof: $R_{0t} \times R_{t0} = \Sigma w_j p_{tj}/p_{0j} \times \Sigma w_j p_{0j}/p_{tj} \geqslant \Sigma w_j = 1$ by the Cauchy-Schwarz inequality. Equality occurs when all p_{tj}/p_{0j} are equal.

10.1.2. If $\boldsymbol{p}_t$ is a permutation of $\boldsymbol{p}_0$ and all $w_j = 1/N$ then $R_{0t} \geqslant 1$.
Proof: See Appendix.

10.1.3. $H' \leqslant G \leqslant R$ with equalities if and only if all price ratios are equal.
Proof: These inequalities are simply the well-known relations between the arithmetic, geometric and harmonic means of positive real numbers.

10.1.4. $H' \leqslant RH \leqslant R$ with equalities if and only if all price ratios are equal.
Proof: This follows from 10.1.3 and the definition of RH, which is $RH = (H'R)^{1/2}$.

10.1.5. $H' \leqslant RA \leqslant R$ with equalities if and only if all price ratios are equal.

Proof: See Appendix. The proof only uses inequality between the arithmetic and the harmonic mean.

Comment to 10.1.1–10.1.5: Note that these inequalities are strictly mathematical, using only the fact that price ratios are positive real numbers.

10.2. Approximations of differences between parameters

By using Taylor expansions around the unit vector (all $r_j = 1$) it is possible to establish approximate relations between those parameters that are functions of price ratios only. It turns out that these relations could be expressed in terms of the following weighted moments of the price ratios $r_j = p_{tj}/p_{0j}$:

$$\mu = \sum w_j r_j$$
$$\sigma^2 = \sum w_j(r_j - \mu)^2 \quad \text{and} \qquad (32)$$
$$\gamma = \sum w_j(r_j - \mu)^3.$$

The accuracy of these approximations depends, of course, on how far away from one the price ratios are, but in most practical situations they could be considered quite reliable. The derivation of these relations is sketched in the Appendix.

The relations are:

10.2.1. $R - G \approx R - RA \approx R - RH \approx \sigma^2/2$ as a result of second-order Taylor approximations.

10.2.2. $R - H' \approx \sigma^2$ as a result of second-order Taylor approximations.

10.2.3. $G - RA \approx \gamma/12 + (\mu - 1)\sigma^2/4$ as a result of third-order Taylor approximations.

Comment to 10.2.1–10.2.3. These approximations show the order of the positive bias of the parameter R in relation to the more accurate parameters G, RA and RH. As could be expected this bias increases with increasing dispersion of the price ratios. 10.2.3 means that G could be expected to be slightly larger than RA in situations of price increases with a positively skewed distribution. This is likely to be a more common case than the contrary.

11. Outcomes of the Swedish CPI with Different Parameters

For January-March and for September 1990 experimental calculations with different parameters were done for the Swedish CPI. Only LOPS and LIPS were involved so the choice of parameter only influenced about 45% of the CPI total. The results for September 1990 with December 1989 as the reference period are presented in Table 1. (They are typical also for the other months.)

The major feature of these results is the size of the differences, which runs contrary to the popular belief that the formula question does not matter very much. With the R parameter the Swedish CPI figure would have been 0.65 percentage units higher than the one presented – 110.63 instead of 109.98. The reasons for these considerable differences are the large variances of the price ratios or, equivalently, the *price bouncing* phenomenon. Price bouncing has three major causes in the Swedish CPI data:

- Seasonal and other variations in the prices of fresh fruits and vegetables.
- Bargain pricing for short periods.
- Substitutions caused by an item disappearing from the market. This is particularly common among clothing items.

Different item groups exhibit different patterns. The parameter differences are largest for fresh food and clothing. The most extreme price bouncing commodity

Table 1. Outcome of the Swedish CPI September 1990 compared to December 1989 according to different parameters in LOPS and LIPS

Item group	Parameter					Weight (%)
	A	R	G	RA	H	
Fresh food	112.62	115.82	112.13	111.92	111.48	3.7
Clothing and shoes	101.01	103.71	100.94	100.98	100.89	7.6
Furniture and household equipment	104.94	105.96	104.68	104.68	104.57	5.2
Articles for recreation	102.97	104.31	102.93	102.93	102.78	4.2
Miscellaneous	112.17	113.29	112.47	112.42	112.75	6.5
LOPS, all items	106.30	108.15	106.23	106.20	106.15	27.4
LIPS, all items	107.02	107.49	106.85	106.85	(105.02)*	18.3
LIPS + LOPS	106.59	107.89	106.48	106.46	(105.70)*	45.7
Difference compared to RA for CPI, all items	+0.06	+0.65	+0.01	0	(−0.35)*	

*The low value is due to a few misprinted items.

was white cabbage, for which the price ratios one month ranged from 0.23 to 7.

The A parameter was used in the Swedish CPI up to 1989. This year a decision was made to use the RA parameter in the long-term index (in a modified version with $p.$ being a mean of twelve months' prices) and R in the short-term index. The reason for this decision was that RA could be motivated by approximation arguments in the long-term index only. The simplicity of R, making it easier to explain to the general public, was the decisive factor in the short-term index since its bias effects were at that time thought to be small. The knowledge of the large differences exhibited in Table 1 led to a change of this decision in May 1990 and from that time on the RA parameter is used in the short-term index as well.

12. Conclusions

When considering what formulas to use in practical index computation it is certainly of importance to consider their performance with regard to index tests. But if a formula does not pass a certain test it is necessary to ask the questions: How large are deviations from ideal performance? and Do deviations go in one direction only or in both directions? The answer to these questions depend partly on results like those in Section 10.2 and partly on actual data.

By looking at 10.2.1 and data like those in Table 1 it is clear that the over-estimating bias of R is a serious one. The failure of RA (and RH) with regard to the permutation test and other tests is, however, not equally serious since G, RA and RH approximate each other to the second order Taylor approximations and since deviations are also very small in practice.

A summary of the properties of the seven index formulas (operational parameters) presented here is firstly that the most popular alternatives - A and R - have such serious defects as to exclude them from serious consideration (unless, of course, we have weights related to quantity; then A should be used.) The same goes for H'. The other four alternatives deserve serious consideration, however. Formula H could be motivated only if definition 5A is accepted and then only for homogeneous item groups which are probably a minority. The other three, G, RH and RA are all functions of the price ratios only (in the case of RA cf. formula (34) below), while as H is not. As demonstrated in Section 10 these three could be expected to give very similar results. Formula G is the winner with regard to index tests but, for the Swedish CPI, RA has the advantage of being a good approximation to a long-term link.

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Appendix

Proof of 10.1.2

We use a theorem in Hardy, Littlewood, and Polya (1934, p. 261) saying that, if $\{a_j\}$ and $\{b_j\}$ are two sets of N real numbers, which are given except in order (arrangement) then Σab is largest when the sets are ordered in the same direction (either both are increasing or both are decreasing) and smallest when the sets are ordered in the opposite direction. Now, setting $\{b_j\} = \{1/a_j\}$ and letting $\{i_j\}$ denote an arbitrary reordering of the elements in the original set, we get

$$\sum_{j=1}^{N} a_j \frac{1}{a_{i_j}} \geqslant \sum_{j=1}^{N} a_j \frac{1}{a_j} = N \tag{33}$$

where $\{a_j\}$ and $\{1/a_j\}$ are arranged in opposite direction on the right hand side of the inequality. Now, setting $a_j = p_{tj}$ and $a_{i_j} = p_{0j}$ the theorem is proved.

Proof of 10.1.5

Setting $r_j = p_{tj}/p_{0j}$ we have, after some algebraic rearrangements:

$$R = \sum w_j r_j;$$

$$RA = \left[\sum w_j(1 + r_j)^{-1}\right]^{-1} - 1;$$

$$H' = \left[\sum w_j r_j^{-1}\right]^{-1}. \tag{34}$$

Applying the harmonic mean - arithmetic mean inequality in 10.1.3 to $\{1 + r_j\}$ and $\{1 + r_j^{-1}\}$, we obtain:

$$RA = \left[\sum w_j(1 + r_j)^{-1}\right]^{-1} - 1$$

$$\leqslant \sum w_j(1 + r_j) - 1$$

$$= \sum w_j r_j = R \tag{35a}$$

$$(H')^{-1} = \sum w_j r_j^{-1}$$

$$= \sum w_j(1 + r_j^{-1}) - 1$$

$$\geqslant \left[\sum w_j(1 + r_j^{-1})^{-1}\right]^{-1} - 1$$

$$= \{\text{after some algebra}\}$$

$$= RA^{-1} \Rightarrow RA \geqslant H'. \tag{35b}$$

Proof of 10.2.1

$$R = R(r_1, \ldots, r_N) = \sum w_j r_j,$$

$$G = G(r_1, \ldots, r_N) = \prod r_j^{w_j},$$

$$RA = RA(r_1, \ldots, r_N) \tag{36}$$

$$= \left[\sum w_j(1 + r_j)^{-1}\right]^{-1} - 1,$$

$$RH = RH(r_1, \ldots, r_N) = \sqrt{\frac{\sum w_j r_j}{\sum w_j / r_j}}.$$

We expand these functions by a Taylor approximation around $\boldsymbol{r}_0 = \mathbf{1} = (1, \ldots, 1)$. In general we have the following second-order approximation, where

$$\mathbf{1} + \boldsymbol{\varepsilon} = (1 + \varepsilon_1, \ldots, 1 + \varepsilon_N):$$

$$f(\boldsymbol{r}) \approx f(\mathbf{1}) + \sum f_j'(\mathbf{1})\varepsilon_j + \tfrac{1}{2}\sum\sum f_{ij}''(\mathbf{1})\varepsilon_i\varepsilon_j.$$

By differentiating the respective functions we get:

$$R(\mathbf{1}) = G(\mathbf{1}) = RA(\mathbf{1})$$

$$= RH(\mathbf{1}) = 1;$$

$$R_k'(\mathbf{1}) = G_k'(\mathbf{1}) = RA_k'(\mathbf{1})$$

$$= RH_k'(\mathbf{1}) = w_k; \tag{37}$$

$$R_{kk}''(\mathbf{1}) = 0;\ G_{kk}''(\mathbf{1}) = RA_{kk}''(\mathbf{1})$$

$$= RH_{kk}''(\mathbf{1}) = -w_k(1 - w_k);$$

For $k \neq 1$ we get:

$$R''_{kl}(\mathbf{1}) = 0; G''_{kl}(\mathbf{1}) = RA''_{kl}(\mathbf{1}) = RH''_{kl}(\mathbf{1}) = w_k w_l;$$

After some algebra the proof is completed.

Proof of 10.2.2

The proof is completely analogous to that of 10.2.1. We have:

$$H' = H'(r_1, \ldots, r_N) = \frac{1}{\sum w_j/r_j};$$

$$H'(\mathbf{1}) = 1; (H')'_k(\mathbf{1}) = w_k;$$

$$(H')''_{kk}(\mathbf{1}) = -2w_k(1 - w_k); \quad (38)$$

For $k \neq 1$ $(H')''_{kl} = 2w_k w_l$.

After some algebra the proof is completed.

Proof of 10.2.3

We use the same symbols as in 10.2.1 above. In general we have the following third-order approximation:

$$f(\mathbf{r}) \approx f(\mathbf{1}) + \sum_j f'_j(\mathbf{1})\varepsilon_j + \tfrac{1}{2}\sum_i\sum_j f''_{ij}(\mathbf{1})\varepsilon_i\varepsilon_j + \tfrac{1}{6}\sum_h\sum_i\sum_j f'''_{hij}(\mathbf{1})\varepsilon_h\varepsilon_i\varepsilon_j. \quad (39)$$

According to 10.2.1 RA and G have the same second-order approximations. By differentiating once more, we get:

$$G'''_{kkk}(\mathbf{1}) = w_k(1 - w_k)(2 - w_k);$$

$$G'''_{kkl}(\mathbf{1}) = -w_k w_l(1 - w_k) \text{ for } k \neq l;$$

$$G'''_{klm}(\mathbf{1}) = w_k w_l w_m$$

for $k \neq l$, $l \neq m$, and $k \neq m$.

$$RA'''_{kkk}(\mathbf{1}) = 1.5w_k(1 - w_k)^2;$$

$$RA'''_{kkl}(\mathbf{1}) = -w_k w_l(1.5w_k - 1)$$

for $k \neq l$;

$$RA'''_{klm}(\mathbf{1}) = 1.5w_k w_l w_m$$

for $k \neq l$, $l \neq m$, and $k \neq m$. (40)

After some algebra the proof is completed.