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Derivation and Properties of the X11ARIMA and Census X11 Linear Filters

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This study extends previous analyses by including *all* the possible combinations of trend-cycle and seasonal filters available in the X11ARIMA with and without extrapolations (in this latter case, they are equivalent to those of X11). The corresponding "cascade" filters for historical and current observations are discussed, including the combined filters for the residuals. It is shown that the variance amplification and phase shift of the standard concurrent filters fall between those corresponding to the combination of longer and shorter trend-cycle and seasonal moving averages. The variance amplification of the shorter filter is the largest whereas the corresponding phase shifts are near zero at low frequencies. The contrary is observed for the longer concurrent filter. The use of ARIMA extrapolations produces gain functions closer to those of the symmetric filters if the ARIMA parameter values are consistent with the filters' implicit assumptions of trend-cycle and seasonality. The presence of inconsistency highly affects the linear properties of the trend-cycle filters.

Key words: Seasonality; trend-cycle; irregular variations; symmetric cascade linear filters; asymmetric cascade linear filters.

1. Introduction

The X11ARIMA seasonal adjustment method, with or without ARIMA extrapolations, is widely applied by statistical agencies. This method developed by Dagum (1980 and 1988) is an extension of the Census X11 method developed by Shiskin, Young, and Musgrave (1967). Both methods apply moving averages or linear smoothing filters to estimate the trend-cycle and seasonal components of a time series.

It is inherent in any moving average procedure that the first and last points of a time series cannot be smoothed with the same symmetric filters applied to middle (central) values. The current and most recent observations are smoothed by asymmetric filters which have different properties concerning the type of functions they reproduce or eliminate.

The linear properties of the seasonal adjustment and trend-cycle symmetric and non-symmetric filters have been already studied for a fixed combination of moving averages

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which is the standard (default) option of the Census X11 method and the 1980 version of X11ARIMA (see, e.g., Young (1968); Wallis (1982); Dagum (1982 and 1983); Burridge and Wallis (1984); Dagum and Laniel (1987) and other references given therein).

Besides the standard (default) moving averages option, the X11ARIMA and X11 methods have a broad range of seasonal and trend-cycle filters the properties of which have not yet been studied. Nevertheless, some of these non-standard filters are frequently used by statistical agencies to seasonally adjust real data. Furthermore, the default option of the last version of X11ARIMA no longer applies a fixed filter combination but a variable combination of seasonal and trend-cycle moving averages based on given values of the signal to noise ratio for each component.

This study discusses the linear properties of *all* possible combinations of seasonal and trend-cycle moving averages available in the X11ARIMA method which includes as a particular case those of the X11 method when ARIMA extrapolations are not used. The resulting cascade filters for the estimation of the seasonal component, the trend-cycle and the irregulars are analysed by means of their gain and phase-shift functions.

Section 2 introduces the definitions of symmetric and asymmetric smoothing linear filters. Section 3 provides the formulas for the central and end-weights trend-cycle moving averages available in the X11ARIMA. Section 4 gives the various formulas for the calculations of the central and end-weights of the seasonal moving averages. Section 5 introduces the cascade filters resulting from the product of different trend-cycle and seasonal single linear filters applied sequentially. Section 6 discusses the linear properties of the cascade filters for seasonal adjustment, trend-cycle and irregulars. It analyses several combinations of symmetric and asymmetric filters, with and without ARIMA extrapolations. Finally, Section 7 gives the conclusions of this study.

2. Smoothing Linear Filters

Given an input series $x_t, t = 1, \ldots, T$, for t sufficiently far removed from the end of the series $(m + 1 \leq t \leq T - m)$ the output value y_t by X11ARIMA and X11 methods are obtained by application of a symmetric filter $h_m(B)$

$$y_t = h_m(B)x_t = \sum_{j=-m}^{m} h_{m,j} x_{t-j} \tag{2.1}$$

where B is the backshift operator defined as $B^m x_t = x_{t-m}$ $(B^0 = 1)$ and $h_{m,j} = h_{m,-j}$. The length of the filter is $2m + 1$.

For current and recent data $(T - m < t \leq T)$ a symmetric filter cannot be applied and, consequently, truncated asymmetric filters $h_j(B)$ are used. For example,

$$y_T^{(0)} = h_0(B)x_T = \sum_{j=0}^{m} h_{0,j} x_{T-j}$$

$$y_{T-k}^{(k)} = h_k(B)x_{T-k} = \sum_{j=-k}^{m} h_{k,j} x_{T-k-j} \tag{2.2}$$

$$y_{T-m}^{(m)} = h_m(B)x_{T-m} = \sum_{j=-m}^{m} h_{m,j} x_{T-m-j}$$

For the filter $h_k(B); k = 0, 1, \ldots, m$ the subscript k indicates the number of values of x_t entering the filter after the observation x_T or the negative of the lower limit of summation in the expression $\Sigma h_{k,j} x_{t-j}$. The filters are time varying in the sense that on running X11ARIMA at time t with original data $x_t, \ldots, x_T$ each of the first and last $m+1$ adjusted values results from $m+1$ *different* filters applied to the input.

When ARIMA extrapolations are used, the asymmetric filters result from the *convolution* of the X11 method smoothing filters with the ARIMA extrapolation filters (see Dagum 1983). The ARIMA forecasts, x^*_{T+k}, can be written as a linear combination of past values, that is,

$$x^*_{T+k} = \sum_{j=0}^{p} \pi_{k,j} x_{T-j} \quad k = 1, 2, \ldots, n \tag{2.3}$$

where $\pi_{k,j}$ denotes the forecast coefficients to be applied to past values of x_T to generate the kth step ahead forecast and n denotes the number of step ahead forecasts, usually equal to 4 or 12 for quarterly and monthly series, respectively. Therefore, the asymmetric combined filter applied to the *last observation* of a series which has been extended with 12 forecasts is

$$h^*_0(B) x_T = \sum_{j=0}^{\max(m,p)} \left(h_{12,j} + \sum_{k=1}^{12} \pi_{k,j} h_{12,-k} \right) x_{T-j} \tag{2.4}$$

The outputs from the X11ARIMA method result from the sequential application or convolution of several individual linear filters to estimate the trend-cycle and seasonal components. These single linear filters are discussed in the next two sections with reference to monthly data only. The extension to quarterly series is straightforward.

3. Individual Trend-cycle Filters

The estimation of the trend-cycle component by the X11ARIMA and X11 methods is made by the application of two different individual linear filters, namely, the 12-term centred moving average (MA) and the Henderson trend-cycle filters.

The 12-term centred MA is applied in the first iteration for a preliminary estimation of the trend and denoted by *D(B)* defined as

$$D(B) = (1/24) B^{-6} (1 + B)(1 + B + B^2 + \ldots + B^{11}) \tag{3.1}$$

The X11ARIMA and X11 methods generate the six missing estimates of the trend-cycle at either end of the series by repeating the first (last) available estimate six times.

The final estimate of the trend-cycle is made by the application of one of three different Henderson filters available in the computer package, namely the 9-, 13-, and 23-term. These filters were developed by Henderson (1916) based on summation formulas mainly used by actuaries. The basic principle for the summation formula is the combination of operations of differencing and summation in such a manner that when differencing above a certain order is ignored, they will reproduce the functions operated on. The merit of this procedure is that the smoothed values thus obtained are functions of a large number of observed values whose errors, to a considerable extent, cancel out. These filters have the properties that, when fitted to second or third degree polynomials, their output will fall exactly on those polynomials and, when fitted to stochastic data, they will give smoother results than can be obtained from the weights which give the middle point of a second degree polynomial fitted by least squares. Recognition of the fact that the smoothness of the resulting filtering depends on the smoothness of the weight diagram led Henderson (1916) to develop a formula which makes the sum of squares of the third differences of the smoothed series a minimum for any number of terms. In other words, the $\Sigma(\Delta^3 y_t)^2$ is minimized (where $\Delta = 1 - B$ is the difference operator, and y_t is the output or smoothed series) if and only if $\Sigma(\Delta^3 h_k)^2$ (where the h_k's are the weights) is minimized (Dagum 1978 and 1985).

If the span of the average is $2m - 3$, Henderson showed that the general expression for the nth term of the filter that minimizes $\Sigma(\Delta^3 h_k)^2$ is

$$\frac{315\{(m-1)^2 - n^2\}\{m^2 - n^2\}\{(m+1)^2 - n^2\}\{(3m^2 - 16) - 11n^2\}}{8m(m^2 - 1)(4m^2 - 1)(4m^2 - 9)(4m^2 - 25)} \tag{3.2}$$

To derive a set of 13 weights from this formula, 8 is substituted for m and the values are obtained for each n from -6 to 6. Thus, the Henderson 13-term trend-cycle symmetric filter is

$$\begin{aligned} H_{13}(B) = {} & -.019B^{-6} - .028B^{-5} + .00B^{-4} + .066B^{-3} \\ & + .147B^{-2} + .214B^{-1} + .240B^{0} + .214B + .147B^{2} \\ & + .066B^{3} + .00B^{4} - .028B^{5} - .019B^{6} \end{aligned} \tag{3.3}$$

The Henderson 9-term symmetric filter is given by (3.2) making $m = 6$, that is

$$\begin{aligned} H_{9}(B) = {} & -.041B^{-4} - .010B^{-3} + .119B^{-2} + .267B^{-1} \\ & + .330B^{0} + .267B + .119B^{2} - .010B^{3} - .041B^{4} \end{aligned} \tag{3.4}$$

Finally, the Henderson 23-term trend-cycle symmetric filter is obtained from (3.2) making $m = 13$, that is

$$\begin{aligned} H_{23}(B) = {} & -.004B^{-11} - .011B^{-10} - .016B^{-9} - .015B^{-8} - .005B^{-7} \\ & + .013B^{-6} + .039B^{-5} + .068B^{-4} + .097B^{-3} + .122B^{-2} + .138B^{-1} \\ & + .144B^{0} + .138B^{1} + .122B^{2} + .097B^{3} + .068B^{4} + .039B^{5} \\ & + .013B^{6} - .005B^{7} - .015B^{8} - .016B^{9} - .011B^{10} - .004B^{11} \end{aligned} \tag{3.5}$$

The calculation of the weights of the asymmetric Henderson filter in the X11ARIMA method is based on the minimization of the mean squared revision (MSR) between the final estimates (obtained by the application of the symmetric filter) and the preliminary estimate (obtained by the application of an asymmetric filter) subject to the constraint that the sum of the weights is equal to one (Laniel 1985, Dagum 1988). The assumption made is that at the end of the series the seasonally adjusted values are equal to a linear trend-cycle plus a purely random irregular $IID \approx (0, \sigma_a^2)$. The equation used in

X11ARIMA is

$$E[r_t^{(i,m)}]^2 = c_1^2\left[t - \sum_{j=-i}^{m} h_{ij}(t-j)\right]^2 + \sigma_a^2 \sum_{j=-m}^{m}(h_{mj} - h_{ij})^2 \tag{3.6}$$

where h_{mj} and h_{ij} are the weights of the symmetric (central) filter and the asymmetric filters, respectively; $h_{ij} = 0$ for $j = -m, \ldots, -i-1, c_1$ is the slope of the line and σ_a^2 denotes the noise variance.

There is a relation between c_1 and σ_a^2 such that the noise to signal ratio, I/C is given by

$$I/C = (4\sigma_a^2/\pi)^{1/2}/|c_1| \tag{3.7}$$

The I/C noise to signal ratio determines the length of the Henderson trend-cycle filter to be applied. Thus, setting $t = 0$ and $m = 6$ for the end weights of the 13-term Henderson, we have

$$\frac{E[r_0^{(i,6)}]^2}{\sigma_a^2} = \frac{4}{\pi(I/C)^2}\left(\sum_{j=-i}^{6} h_{ij}\right)^2 + \sum_{j=-6}^{6}(h_{6j} - h_{ij})^2 \tag{3.8}$$

Making $I/C = 3.5$ (the most noisy situation where the 13-term Henderson is applied), equation (3.6) gives the same set of end weights of the X11 method. The end weights for the remaining monthly Henderson filters are calculated using $I/C = .99$ for the 9-term filter and $I/C = 7$ for the 23-term filter.

4. Individual Seasonal Filters

The seasonal filters are applied to the seasonal-irregular ratios (differences) of each month separately over a period ranging from 3 to 11 years in order to estimate the seasonal component. The weights are all positive and, consequently, they reproduce exactly the middle value of a straight line within their spans. This property enables the X11ARIMA and X11 programs to estimate a linearly moving seasonality within three- and eleven-year spans. Therefore, these filters can approximate quite adequately gradual seasonal changes that follow non-linear patterns over the whole range of the series.

The seasonal filters of X11ARIMA and X11 enable the estimation of different seasonal patterns, the greater the movement in the seasonal component the shorter the filter choice. If the seasonal component appears very stable, then it can be best estimated by an unweighted average of the seasonal-irregular ratios over the whole span of the series.

The seasonal filters available are: (1) a weighted 3-term MA (3×1); (2) a weighted 5-term MA (3×3); (3) a weighted 7-term MA (3×5); (4) a weighted 11-term MA (3×9); and (5) unweighted simple average.

The seasonal linear filters applied to central values are symmetric and calculated as follows

$$S_{3xk}(B) = (3k)^{-1}B^{-12[1+(k-1)/2]}(1 + B^{12} + B^{24})(1 + \ldots + B^{(k-1)12}); \quad k = 1, 3, 5, 9 \tag{4.1}$$

and

$$S_N(B) = (1/N)(1 + B^{12} + B^{24} + B^{36} + \ldots + B^{12N}) \qquad (4.2)$$

(unweighted simple average; N = number of years)

The X11ARIMA and X11 methods also enable the application of *different* seasonal filters for each month, e.g., a 3×3 MA for the months of June, July, August and a 3×5 MA for the remaining months.

The asymmetric filters applied to non-central values are given in Appendix.

5. Cascade Linear Filters

The final estimates of each component as well as the seasonally adjusted series from X11ARIMA and X11 are obtained by cascade filtering that results from the convolution of the various individual linear filters discussed in previous sections and which are applied sequentially. In fact, if the output from the filtering operation H is the input to the filtering operation Q, the coefficients of the cascade filter C result from the product of $H \times Q$. For symmetric filters $H \times Q$ equals $Q \times H$ but this is not valid for asymmetric filters.

The X11ARIMA and X11 *Seasonal Cascade Filter* most often applied results from the convolution of: (i) 12-term centred MA; (ii) 3×3 MA; (iii) 3×5 MA; and (iv) 13-term Henderson MA.

In symbols

$$S = D^c S_{3\times5}[H_{13}(D^c S_{3\times3} D^c)^c]^c \qquad (5.1)$$

For *central* observations, D, H_{13}, $S_{3\times3}$ and $S_{3\times5}$ are defined by equations (3.1), (3.3), (4.1) respectively and where c denotes the complement of the corresponding filter. For *end-values*, D, H_{13}, $S_{3\times3}$ and $S_{3\times5}$ are defined by equations (3.1), (3.8), A.2 and A.3, respectively.

The complement of (5.1) defines the corresponding *Seasonal Adjustment Cascade Filter*, that is $S^c = I - S$, where I denotes the identity filter.

The *Trend-Cycle Cascade Filter* most often applied is

$$TC = H_{13}\{I - D^c S_{3\times5}[H_{13}(D^c S_{3\times3} D^c)^c]^c\} \qquad (5.2)$$

and the *Irregular Component Cascade Filter* is given by the complement of the Trend-Cycle Cascade Filter, that is $IR = I - S - TC = S^c - TC$.

6. Properties of the X11ARIMA and X11 Cascade Linear Filters

The properties of the cascade filters can be studied by analysing their corresponding frequency response functions.

The frequency response function is defined by

$$H(\omega) = \sum_{j=-m}^{m} h_j e^{-i2\Pi\omega j}; \quad 0 \le \omega \le \tfrac{1}{2} \qquad (6.1)$$

where h_j are the weights of the filter and ω is the frequency in cycles per unit of time.

In general, the *frequency response* function can be expressed in polar form as follows

$$H(\omega) = A(\omega) + iB(\omega) = G(\omega)e^{i\phi(\omega)} \qquad (6.2)$$

where $G(\omega) = \{A^2(\omega) + B^2(\omega)\}^{1/2}$ is called the *gain* of the filter and $\phi(\omega) = \arctan\{-B(\omega)/A(\omega)\}$ is called the *phase shift* of the filter and is usually expressed in radians. Expression (6.2) shows that if the input function is a sinusoidal variation of unit amplitude and constant phase shift $\psi(\omega)$, the output function will also be sinusoidal but of amplitude $G(\omega)$ and phase shift $\psi(\omega) + \phi(\omega)$. The gain and phase shift vary with ω. For symmetric filters the phase shift is 0 or $\pm$ Π, and for asymmetric filters take values between $\pm$ Π at those frequencies where the gain function is not zero. For a better interpretation, the phase shifts will be given here in months instead of radians (the phase shift in months is given by $\phi(\omega)/2\Pi\omega$ for $\omega \neq 0$).

The gain functions shown below should be interpreted as relating the spectrum of the original series to the spectrum of the output obtained with a linear time invariant filter. For example, let $y_t^{(0)}$ be the estimated seasonally adjusted observation for the current period based on data $x_t, t = 1, 2, \ldots, T$, then the time series $\{y_t^{(0)}\}$ is obtained from $\{x_t\}$ by application of the concurrent linear time invariant filter $h^{(0)}(B)$. The gain functions discussed below relate the spectrum of $\{x_t\}$ to the spectrum of $\{y_t^{(0)}\}$ not to the spectrum of the complete seasonally adjusted series produced at time t (which includes $y_t^{(0)}$, a first revision of time $t - 1$, a second revision of time $t - 2$, and so on).

6.1. *Properties of symmetric (central) filters*

The gain functions of three different seasonal adjustment cascade filters are shown in Figure 1. The gain function of the cascade filter resulting from the sequential combination or product of short moving averages, i.e., $(3 \times 3)(3 \times 3)$ [H-9] has broader dips around the fundamental seasonal frequency $\omega = 0.083$ and its five harmonics 0.167, 0.250, 0.330, 0.417, and 0.50. Therefore, this combination is more appropriate for series affected by rapidly changing seasonality.

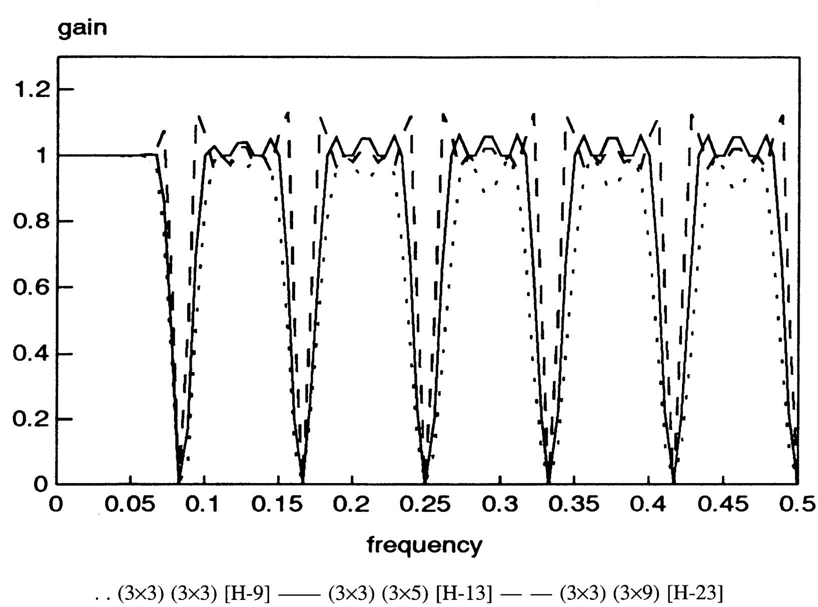


Fig. 1. Seasonal adjustment symmetric cascade filters

On the other hand, the gain function of the cascade filter corresponding to the product of long filters, i.e., (3 × 3)(3 × 9)[H-23] shows narrower seasonal dips, and is therefore more appropriate for series with an underlying stable or regular seasonal pattern. The gain of the cascade filter for the standard combination (3 × 3)(3 × 5)[H-13] falls between that of the short and the long filters. It will be seen that this middle position for the standard cascade filter is always present irrespective of the type of individual filters sequentially combined.

Figure 2 shows the gains of the trend-cycle cascade filters for the same sequential combinations discussed above. The short and standard cascade filters modify slightly the variances of the low frequency components; i.e., $0 < \omega \leq 0.055$ which corresponds to cycles of periodicities equal to and longer than 18 months.

On the other hand, the long cascade filter reflects the pattern of the long Henderson filter [H-23] with a gain function that converges fast to zero at the fundamental seasonal frequency and oscillates around zero at the remaining higher frequencies.

The total variance of the trend-cycle cascade filter as shown by the area under the gain function clearly indicates that it is smallest for the long combination which makes this combination suitable for series with an underlying stable (more rigid) trend-cycle.

On the other hand, the short cascade filter passes about 90% of the variance associated with the frequency band $0.08 < \omega < 0.16$ and 25% for $0.16 < \omega < 0.25$, which makes it more appropriate for series affected by rapidly changing trend-cycle variations.

Finally, Figure 3 gives the gain of the cascade filters for the irregular component. As shown, the area for the long cascade filter is largest, indicating the variance of the estimated residuals will be larger relative to that given by the two other cascade filters. In fact, assuming that the irregulars are white noise with $\sigma_I^2 = 1$, then the variance of the residuals is given by $\sigma_R^2 = \sigma_I^2 \Sigma_{j=-m}^{m} h_j^2$, where h_j are the weights of the cascade filter. For the long cascade filter $\sigma_R^2 = 0.73$, whereas for the short and standard cascade filter

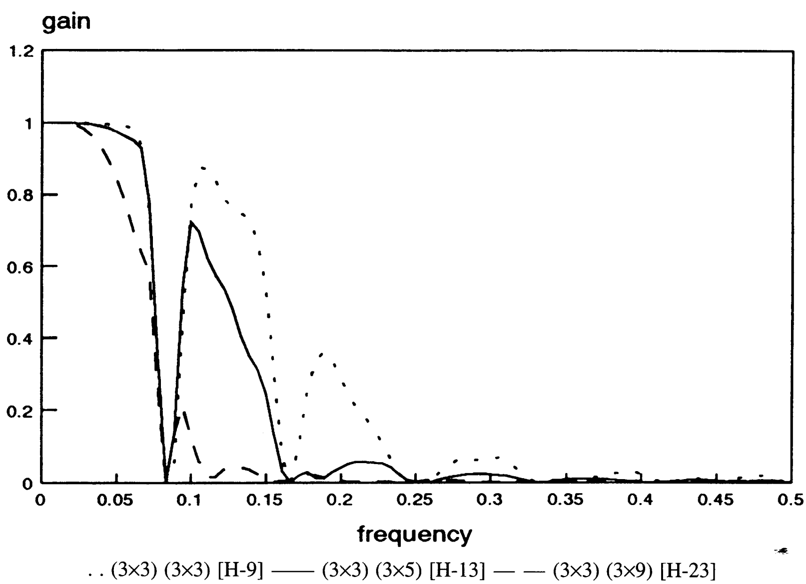


Fig. 2. Trend-cycle symmetric cascade filters

they are $\sigma_R^2 = 0.36$ and $\sigma_R^2 = 0.55$, respectively (Dagum, Chhab, and Solomon 1991). From the shape of the gain function we can also infer the presence of negative autocorrelations, particularly at low and seasonal lags. The broader the seasonal dips are, the larger the negative autocorrelations. Assuming in (2.1) that input x_t is white noise, the autocorrelation function of the y_t output is given by

$$\rho_{YY}(k) = \frac{\sum_{j=-m}^{m} h_j h_{j+k}}{\sum_{j=-m}^{m} h_j^2}$$

For the standard option, the autocorrelations of the residuals are as follows

$$\rho_1 = -0.34 \;\; \rho_2 = -0.21 \;\; \rho_3 = -0.06 \;\; \rho_4 = 0.05 \;\; \rho_5 = 0.08 \;\; \rho_6 = 0.02 \;\; \rho_7 = -0.05$$

$$\rho_8 = -0.03 \;\; \rho_9 = 0.02 \;\; \rho_{10} = 0.07 \;\; \rho_{11} = 0.11 \;\; \rho_{12} = -0.32 \;\; \rho_{13} = 0.11$$

For the short cascade filter the autocorrelations of the residuals show a similar pattern but are higher at lags 1 and 12.

$$\rho_1 = -0.47 \;\; \rho_2 = -0.17 \;\; \rho_3 = 0.08 \;\; \rho_4 = 0.10 \;\; \rho_5 = -0.03 \;\; \rho_6 = -0.01 \;\; \rho_7 = 0.01$$

$$\rho_8 = -0.04 \;\; \rho_9 = -0.04 \;\; \rho_{10} = 0.07 \;\; \rho_{11} = 0.20 \;\; \rho_{12} = -0.43 \;\; \rho_{13} = 0.21$$

For the long cascade filter, the autocorrelations of the residuals are smaller at lags 1 and 12 than those of the previous combinations. The autocorrelations follow

$$\rho_1 = -0.19 \;\; \rho_2 = -0.17 \;\; \rho_3 = -0.13 \;\; \rho_4 = -0.08 \;\; \rho_5 = -0.04 \;\; \rho_6 = 0.00 \;\; \rho_7 = 0.03$$

$$\rho_8 = 0.05 \;\; \rho_9 = 0.05 \;\; \rho_{10} = 0.04 \;\; \rho_{11} = 0.03 \;\; \rho_{12} = -0.15 \;\; \rho_{13} = 0.02$$

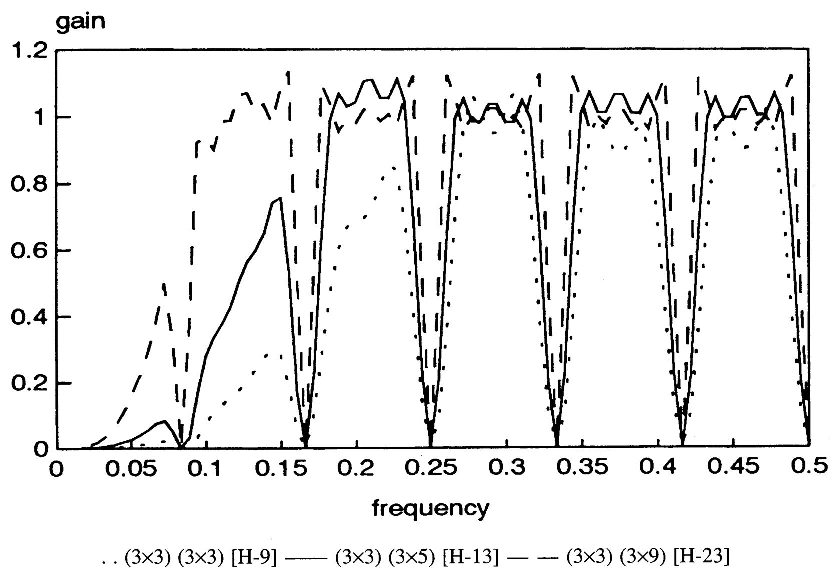


Fig. 3. Symmetric cascade filters for the irregular component

6.2. Properties of the asymmetric cascade filters

6.2.1. Concurrent cascade filters without ARIMA extrapolations

To analyse the asymmetric concurrent cascade filters for seasonal adjustment (the concurrent filter is applied to the last available observation) when ARIMA extrapolations are not used, we selected three sequential combinations as follows:

(1.a) Short seasonal moving averages combined with each of the three Henderson filters; that is, $(3\times3)(3\times3)$[H-9], $(3\times3)(3\times3)$[H-13] and $(3\times3)(3\times3)$[H-23];

(2.a) Standard seasonal moving averages combined with each of the three Henderson filters, that is, $(3\times3)(3\times5)$[H-9]; $(3\times3)(3\times5)$[H-13] and $(3\times3)(3\times5)$[H-23]

and

(3.a) Long seasonal moving averages combined with each of the three Henderson filters, that is, $(3\times3)(3\times9)$[H-9], $(3\times3)(3\times9)$[H-13] and $(3\times3)(3\times9)$[H-23].

Figures 4, 5, and 6 give the corresponding gain and phase shift functions for each case.

We can see that if the seasonal filters are short (Figure 4), the short Henderson filter amplifies very much the variance at frequencies near the fundamental seasonal as well as those frequencies between the fundamental seasonal and its first harmonic. On the other hand, short seasonal filters combined with the long Henderson do not amplify the gain but increase the phase shift.

As the seasonal filters become longer from (3×3) to (3×5) to (3×9), the influence of the Henderson filters diminishes. In fact, as plotted in Figure 6 the combination $(3\times3)(3\times9)$ [H-9] has reduced very much the amplification observed in the short cascade filter. Furthermore, it has also improved the results from the other two combinations, i.e., using [H-13] and [H-23] by reducing the variance amplification and also the phase shift for the combination with [H-23].

All of the above seems to indicate that if ARIMA extrapolations are not used, the longer seasonal moving averages are to be preferred for concurrent seasonal adjustment, at least, from the view point of the linear properties of the X11ARIMA method.

Combinations (1.a), (2.a) and (3.a) were also applied to obtain the gain and phase shifts of the concurrent trend-cycle and the concurrent irregular cascade filters, (not shown here for space reasons).

The conclusions obtained are summarized as follows:

The Henderson filters significantly increase the variance and phase shift of the concurrent trend-cycle cascade filters at the frequencies near the fundamental seasonal and between the fundamental seasonal and its first harmonic. This would provide a justification to the common practise among statistical agencies of not publishing trend estimates of current observations because they are often subject to large revisions.

The trend-cycle concurrent cascade filter passes a significant amount of noise at high frequencies. The relationship between the Henderson filter's length and the trend-cycle cascade filter's length is similar to that observed for the seasonal adjustment filters although less pronounced, i.e., the longer the cascade filter the lesser the effect of the Henderson filter.

Concerning the concurrent irregular cascade filters, they show a larger variance than those corresponding to the symmetric filters at low frequencies and between the fundamental seasonal and its first harmonic but smaller variance at the remaining frequencies.

Among the various irregular concurrent filters, the variance is largest for the long cascade filter which agrees with the fact that its complement, the trend-cycle concurrent

Fig. 4. SEASONAL ADJUSTMENT CONCURRENT CASCADE FILTER
Short seasonal MA combined with three Henderson filters

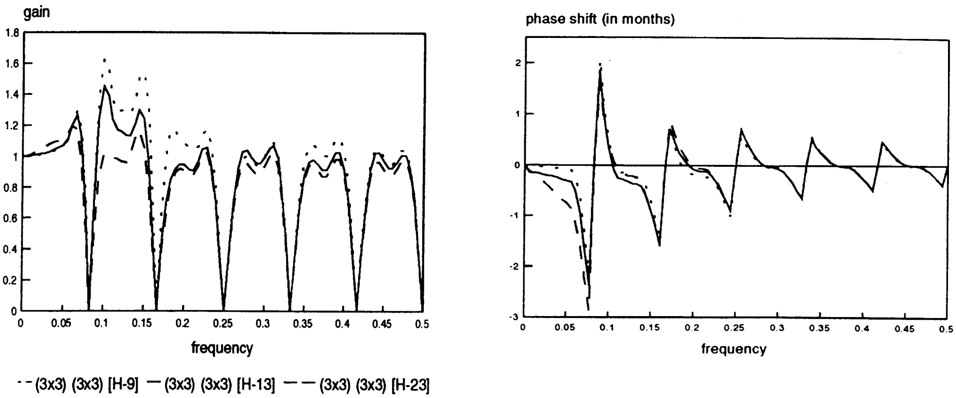


Fig. 5. SEASONAL ADJUSTMENT CONCURRENT CASCADE FILTER
Standard seasonal MA combined with three Henderson filters

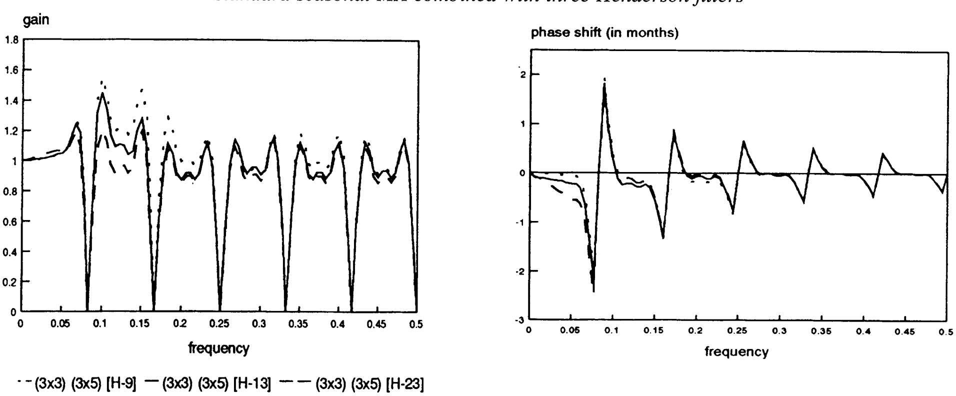
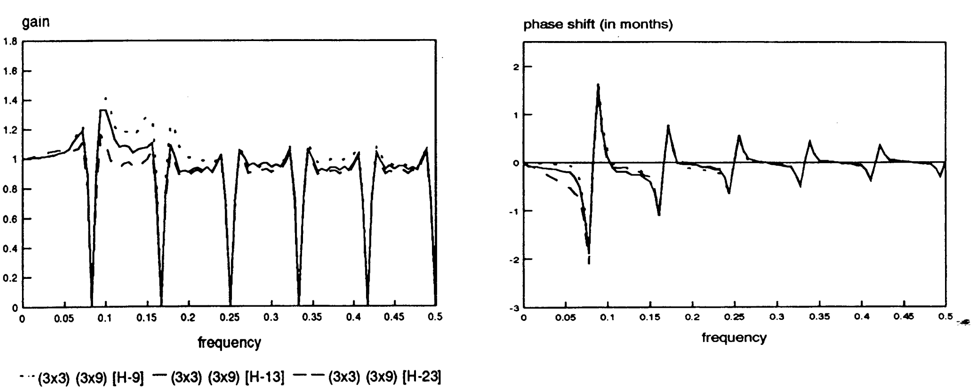


Fig. 6. SEASONAL ADJUSTMENT CONCURRENT CASCADE FILTER
Long seasonal MA combined with three Henderson filters



filter, smooths more than any other combination. The opposite is observed for the short irregular cascade filter.

6.2.2. Concurrent cascade filters with consistent and inconsistent ARIMA extrapolations
To discuss the effect of the ARIMA extrapolations on the concurrent cascade filters analysed in the previous section we select the following sequential combinations:

(1.b) Standard cascade filter $(3\times3)(3\times5)$[H-13] extended with ARIMA forecasts from a (0,1,1)(0,1,1) model with parameter $\theta = .40$ $\Theta = .60$.
(2.b) Short cascade filter $(3\times3)(3\times3)$[H-9] extended with the same ARIMA model as in (1) but $\theta = .30$ $\Theta = .30$.
(3.b) Short cascade filter $(3\times3)(3\times3)$[H-9] extended with the same ARIMA model as in (1) but $\theta = .80$ $\Theta = .80$.
(4.b) Long cascade filter $(3\times3)(3\times5)$[H-23] extended with the same ARIMA model as in (1) but $\theta = .80$ $\Theta = .80$.
(5.b) Long cascade filter $(3\times3)(3\times5)$[H-23] extended with the same ARIMA model as in (1) but $\theta = .30$ $\Theta = .30$.

The $(0,1,1)(0,1,1)_{12}$ ARIMA model may be expressed by

$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t \qquad (6.2.1)$$

with invertibility conditions $|\theta| < 1$ and $|\Theta| < 1$ (Box and Jenkins 1970).

The extrapolation filter of the (0,1,1) (0,1,1) ARIMA model implies an instantaneously straight-line trend and an instantaneously constant zero-sum seasonal pattern, both changing their level and slope in a stochastic fashion, proportional to the value of the innovation a.

The parameters θ and Θ can be interpreted as representing the extent to which the trend level and the seasonal pattern respond to new innovations. Thus, a low value of θ corresponds to a fast-changing trend, and a high value to an underlying stable trend. Similarly, a low value of Θ corresponds to a rapidly changing seasonality, and a high value to a stable underlying seasonal pattern.

The sets of parameter values chosen are those discussed by Dagum (1983) which have been observed in empirical cases. The parameter values for combination (1.b) correspond to the classical international airline passengers series discussed by Box and Jenkins (1970), those of combination (2.b) are often encountered in some retail trade series and combination (3.b) can be found in industrial employment data.

For the combinations (1.b), (2.b) and (4.b) the ARIMA parameter values are consistent with the assumptions implied by the corresponding symmetric cascade filters. For example, the short combination will be appropriate for a fast changing trend-cycle and fast changing seasonality which agrees with the low parameter values of the ARIMA model. Similarly, the (4.b) combination is appropriate for the more stable trend-cycle and seasonality. But for the case of $\Theta > 0.80$ the use of the 3×9 seasonal filter should be preferred. On the other hand, combinations (3.b) and (5.b) are selected to assess the effect of using extrapolations from an ARIMA model *not consistent* with the assumptions of the corresponding symmetric cascade filter.

Figures 7, 8, and 9 exhibit the gains and phase shifts of the three components, with and without extrapolations, for combination (1.b). It can be seen that the effects of the ARIMA extrapolations on the gain functions are: (1) a significant reduction of variance amplification at low frequencies and between the fundamental seasonal and its first harmonic; and (2) broader seasonal dips.

As a consequence, the gain functions of the seasonal adjustment and trend-cycle filters

CONCURRENT STANDARD CASCADE FILTERS WITH CONSISTENT AND NO ARIMA EXTRAPOLATIONS

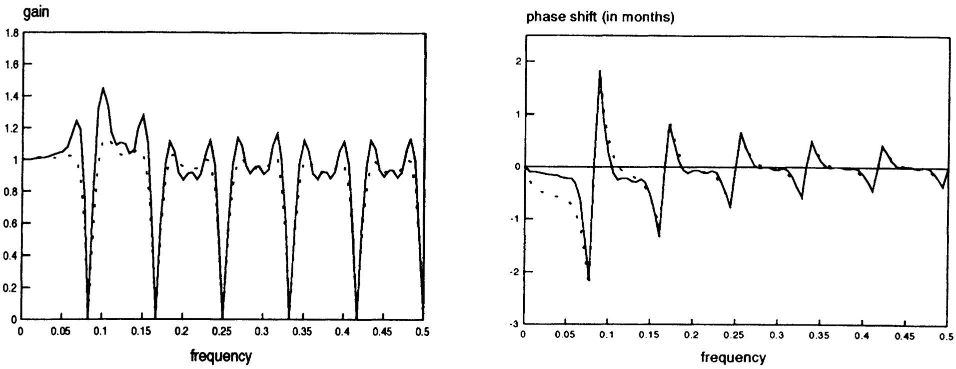


Fig. 7. Seasonal adjustment

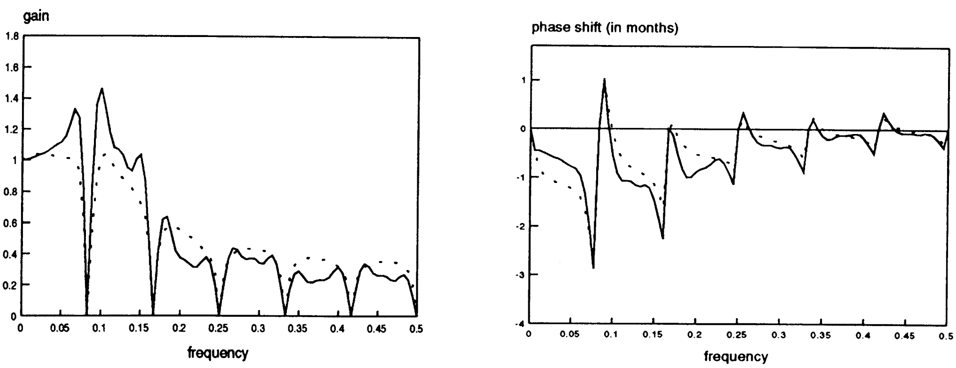


Fig. 8. Trend-cycle

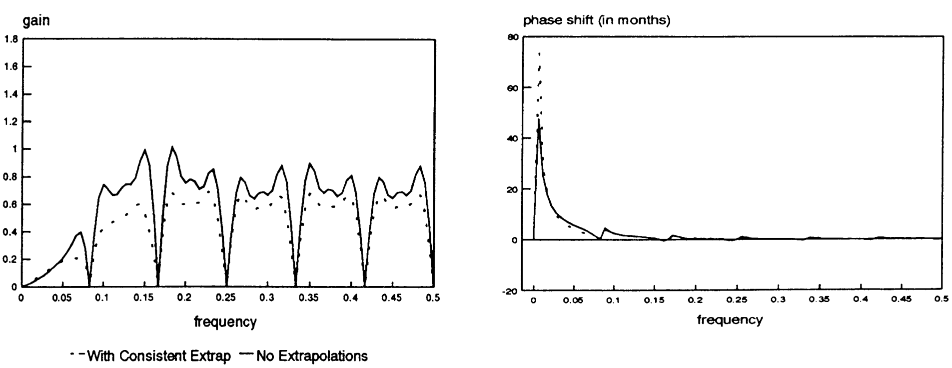


Fig. 9. Irregular

resemble more those of the corresponding symmetric cascade filters. Since the concurrent trend-cycle filter still passes a significant amount of noise, the gain of the irregular concurrent filter has smaller variance compared to the corresponding symmetric filter. As per the phase shift, there is a systematic increase of about one month at the low frequencies.

CONCURRENT SHORT CASCADE FILTERS WITH CONSISTENT, INCONSISTENT AND NO ARIMA EXTRAPOLATIONS

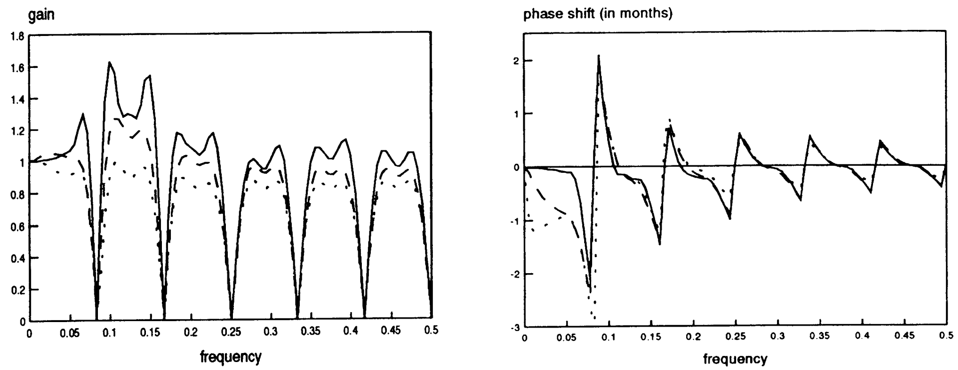


Fig. 10. Seasonal adjustment

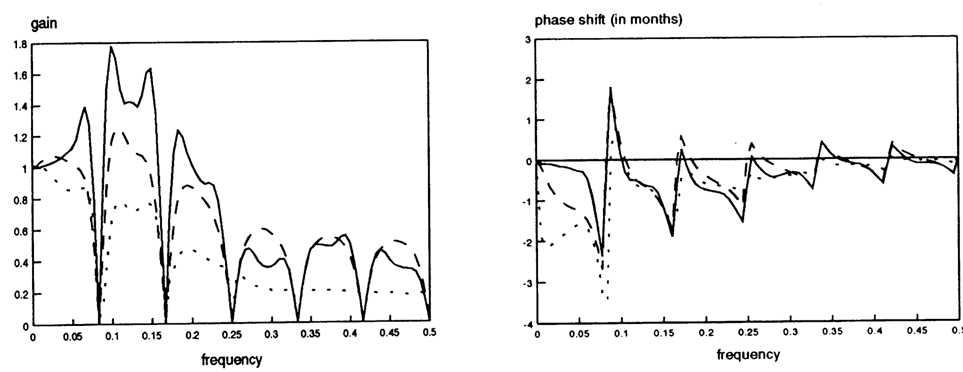


Fig. 11. Trend-cycle

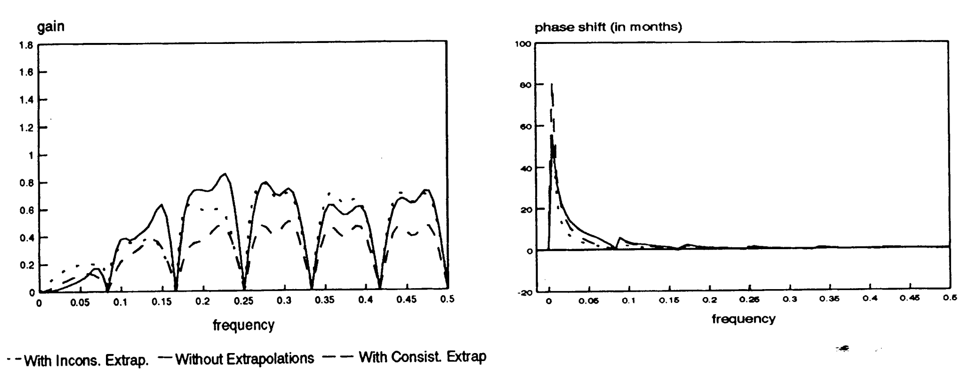


Fig. 12. Irregular

All the above observations are also valid for combinations (2.b), (4.b) and, in general, whenever the ARIMA parameter values are consistent with the implicit assumptions of the corresponding symmetric cascade filters.

The gain and phase shift functions of the short cascade filters extended with consistent

CONCURRENT LONG CASCADE FILTERS WITH CONSISTENT, INCONSISTENT AND NO ARIMA EXTRAPOLATIONS

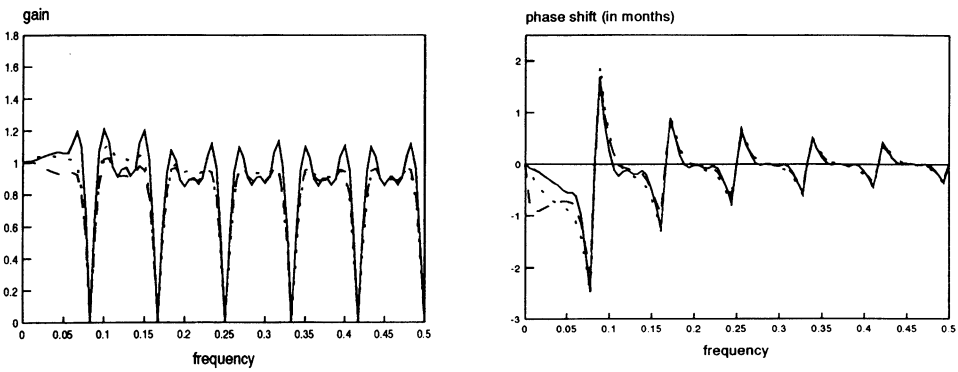


Fig. 13. Seasonal adjustment

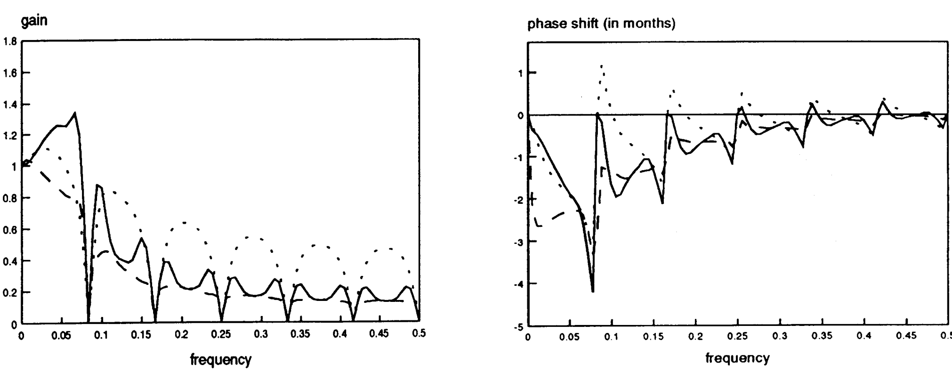


Fig. 14. Trend-cycle

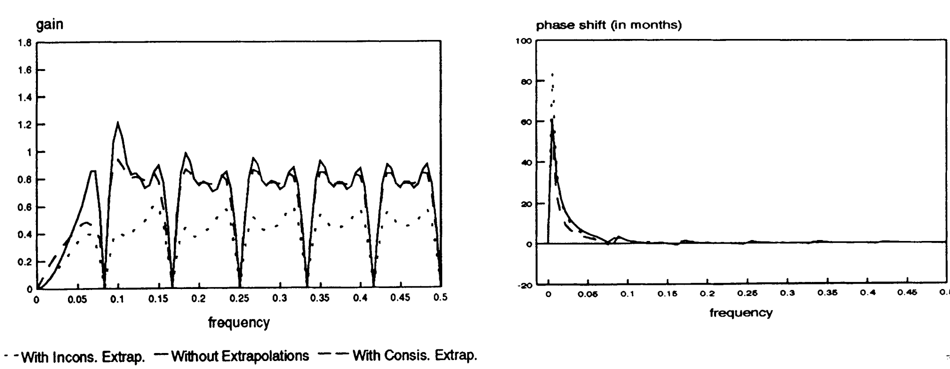


Fig. 15. Irregular

and inconsistent ARIMA extrapolations, combinations (2.b) and (3.b), respectively, are plotted in Figures 10–12.

Figures 10 and 11 show a major reduction of variance over all frequencies when the filter is extended with inconsistent extrapolations. This is attributed to the high value of the trend-cycle parameter θ which plays a crucial role in determining the shape of the gain function. On the other hand, a high value of Θ does not affect the seasonal dips which continue to be broader as in combinations (1.b) and (2.b).

We can also observe that the filters extended with inconsistent ARIMA extrapolations show a substantial increase in phase shift at low frequencies, which is larger for the trend-cycle cascade filter than the seasonal adjustment one. Finally, Figure 12 shows the distortion introduced in the gain function of the irregular, mainly at low frequencies.

Figures 13–15 exhibit the gain and phase shift functions of combinations (4.b) and (5.b). The effect of the a $\theta = 0.30$ is clearly seen in Figure 14 where the variance and phase shift of the trend-cycle filter extended with inconsistent extrapolations are greatly increased over all frequencies.

On the contrary, the long seasonal adjustment concurrent filter (Figure 13) seems to be less distorted than the trend-cycle cascade ones when the extrapolation model parameters are inconsistent.

Finally, the irregular concurrent cascade filter (Figure 15) shows a decrease of variance over all the frequencies together with a high increase in phase shift.

Although not discussed in this study, the discrepancies between the asymmetric and symmetric cascade filters are related to the problem of revision of seasonally adjusted data. Longer asymmetric filters have gain functions that change little as they approach the symmetric ones, an indication that the seasonally adjusted values will be slightly modified when some observations are added to the series, but the convergence of the preliminary estimates to the final values takes many years. The opposite can be concluded for short filters.

7. Conclusions

We have introduced and analysed the cascade filters for seasonal adjustment, trend-cycle estimation and the estimated irregulars (residuals) resulting from the convolution of: (a) very short seasonal and trend-cycle moving averages, $(3\times3)(3\times3)$[H-9]; (b) standard (most frequently applied) seasonal and trend-cycle moving averages, $(3\times3)(3\times5)$ [H-13] and (c) long seasonal and trend-cycle moving averages, $(3\times3)(3\times9)$[H-23].

The short symmetric seasonal adjustment filter is characterized by broad seasonal dips which make it more appropriate for series affected by rapidly changing seasonality. The short trend-cycle cascade filter also seems more adequate for fast changing cyclical variations; its gain function passes all the variance at low frequencies and about 90% of the power at frequencies between the fundamental seasonal and its first harmonic.

The gain function of the short irregular cascade filter exhibits small variance at all frequencies and its shape indicates the presence of negative autocorrelations at low and seasonal lags.

The opposite is concluded by looking at the gain functions of the long symmetric cascade filters. These filter convolutions seem to fit better series with underlying regular trend and stable seasonality. The gain of the corresponding irregular cascade filter shows larger variance and lower autocorrelations at low and seasonal lags. Finally, the gains of the standard symmetric cascade filters fall between the above two cases.

We also analysed the asymmetric cascade filters applied to the last available point (also known as the concurrent filter), with and without ARIMA extrapolations (in this latter case, the filters are equivalent to those of the Census X11 variant). If there are no ARIMA extrapolations, the phase shift for the short concurrent cascade filter is nearly zero at low frequencies but the gain function is highly amplified. On the contrary, the convolution of the long filters shows phase shifts of about one month at low frequencies and practically no amplification of variance at these frequencies. Finally, the phase shift and variance amplification produced by the concurrent standard cascade filter fall between the above two cases.

To analyse the effect of the ARIMA extrapolations on the concurrent cascade filters we discussed three cases where the ARIMA parameter values conform to the assumptions implied by the corresponding symmetric cascade filters and two opposite cases.

The results show that the use of extrapolated values highly improve the gain functions of the concurrent cascade filters if the parameter values of the ARIMA extrapolation model are consistent with the assumptions of the corresponding symmetric cascade filters.

On the other hand, extrapolations from ARIMA models where the parameter values are inconsistent with the implicit assumptions of the symmetric filters produce contradictory results summarized as follows:

(a) the gains of the short cascade filters are mainly affected by large values of θ and practically nothing by Θ. Furthermore, the phase shifts are increased up to two months for the concurrent trend-cycle filter;
(b) the gain and phase shift of the long seasonal adjustment cascade filter are slightly affected by low values of θ and Θ whereas those for the trend-cycle are completely distorted.

In general, ARIMA parameter values not consistent with the symmetric cascade filters mainly affect the trend-cycle gain and phase shift functions, an indication that an inadequate value of θ has greater effect than that of Θ. The inconsistency slightly affects the gain function of the seasonal adjustment filter but increases, in general, the phase shift at low frequencies.

Appendix

Asymmetric Seasonal Filters

Asymmetric Filter of the 3×1 MA

$$S^0_{3\times1}(B) = (.61 + .39B^{12}) \qquad \text{A.1}$$

Asymmetric Filters of the 3×3 MA

$$S^0_{3\times3}(B) = .407(1 + B^{12}) + .185B^{24}$$

$$S^1_{3\times3}(B) = .259(B^{-12} + B^{12}) + .370 + .111B^{24} \qquad \text{A.2}$$

Asymmetric Filters of the 3×5 MA

$$S^0_{3\times5}(B) = .283(1 + B^{12} + B^{24}) + .150B^{36}$$

$$S^1_{3\times5}(B) = .250(B^{-12} + 1 + B^{12}) + .183B^{24} + .067B^{36} \quad \text{A.3}$$

$$S^2_{3\times5}(B) = .150B^{-24} + .217(B^{-12} + 1 + B^{12}) + .133B^{24} + .067B^{36}$$

Asymmetric Filters of the 3×9 MA

$$S^0_{3\times9}(B) = .246 + .221B^{12} + .197B^{24} + .173B^{36} + .112B^{48} + .051B^{60}$$

$$S^1_{3\times9}(B) = .208B^{-12} + .192 + .176B^{12} + .160B^{24} + .144B^{36} + .092B^{48} + .028B^{60}$$

$$S^2_{3\times9}(B) = .173B^{-24} + .163B^{-12} + .154 + .143B^{12} + .133B^{24} + .123B^{36} + .079B^{48} + .032B^{60}$$

$$S^3_{3\times9}(B) = .141B^{-36} + .137B^{-24} + .132B^{-12} + .128 + .123B^{12} + .117B^{24} + .113B^{36} + .075B^{48} + .034B^{60}$$

$$S^4_{3\times9}(B) = .084B^{-48} + .120B^{-36} + .118B^{-24} + .117B^{-12} + .116 + 114B^{12} + .113B^{24} + .111B^{36} + .073B^{48} + .034B^{60}$$

A.4

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