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Erratum

The article "Small Area Estimation via Generalized Linear Models" by Alasdair Noble, Stephen Haslett, and Greg Arnold published in Volume 18, Number 1, 2002, pp. 45–60.

This article above contained a small but central error:
Equation 3.2 on p. 48 should be

$$\boldsymbol{Y}_0 = g^{-1}(\boldsymbol{X\beta}) + \boldsymbol{e}$$

from which it follows that

$$g_0(\boldsymbol{Y}_0) = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$$

for some data-dependent composite function g_0.

The equation on the line preceding Equation 3.3 should be $\boldsymbol{Y} = g_0(\boldsymbol{Y}_0)$.

As it stands there is a possible confusion, in that the model represented by Equation 3.2 is a model in which the data has been transformed rather than a generalized linear model.

A more complete explanation is given below:

In generalized linear models, $\boldsymbol{X\beta}$ is transformed by an inverse link function g^{-1} (Nelder and Wedderburn 1972) so that $\boldsymbol{Y}_0 = g^{-1}(\boldsymbol{X\beta}) + \boldsymbol{e}$. It follows that $\boldsymbol{Y} = g_0(\boldsymbol{Y}_0)$ for some g_0 where $\boldsymbol{Y}_0$ are the original observations. Where $g(\boldsymbol{Y}_0)$ is well defined, as it is for example for loglinear models without zero counts, $g \approx g_0$.

As Nelder and Wedderburn (1972) and del Pino (1989) outline, one method of solution for generalized linear models is via iterated generalized least squares (IGLS), so that at each iteration a linear model is fitted that has the *same* design matrix $\boldsymbol{X}$ but a different (estimated) variance covariance matrix $\boldsymbol{V}$. At covergence then, as for any other iteration, the final fit in an IGLS algorithm involves fitting a linear model with an estimated $\boldsymbol{V}$. As a consequence we must conclude that, *regardless of the fitting procedure*, the final solution is equivalent to fitting a particular linear model, namely $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ that has the same design matrix $\boldsymbol{X}$ as in the more usual specification $\boldsymbol{Y}_0 = g^{-1}(\boldsymbol{X\beta}) + \boldsymbol{e}$ but uses the data dependent function $\boldsymbol{Y} = g_0(\boldsymbol{Y}_0)$.

If this model is required formally it can be found, with the corresponding $\boldsymbol{V}$, via the IGLS algorithm. At each step q in the iteration for fitting a generalized linear model using IGLS, a linear model is fitted with new observation vector and covariance but the same $\boldsymbol{X}$, i.e., at each step the least squares solution is used. At any iteration q,

$$\boldsymbol{Y}^{(\mathrm{q})} = \boldsymbol{X}\hat{\boldsymbol{\beta}}^{(\mathrm{q}-1)} - \boldsymbol{G}^{(\mathrm{q}-1)}(\boldsymbol{Y}_0 - \boldsymbol{\mu}^{(\mathrm{q}-1)})$$

and

$$\mathrm{var}\,(\boldsymbol{Y}^{(\mathrm{q})}) = \boldsymbol{G}^{(\mathrm{q}-1)}\boldsymbol{V}_{\boldsymbol{Y}}\boldsymbol{G}^{(\mathrm{q}-1)^{\mathrm{T}}}$$

with

$$G^{(q-1)} = \left[\frac{\boldsymbol{\delta}(X\boldsymbol{\beta})}{\boldsymbol{\delta\mu}}\right]^{(q-1)}_{n\times n}$$

where $\boldsymbol{Y}_0$ is the original data, and $\boldsymbol{V}_{Y_0} = \mathrm{var}(\boldsymbol{Y}_0)$

To get the explicit relationship between $\boldsymbol{Y}_0$ and $\boldsymbol{Y}^{(q)}$, note that at convergence

$$\boldsymbol{Y}^{(q)} = \boldsymbol{X}\hat{\boldsymbol{\beta}}^{(q)} - \boldsymbol{G}^{(q)}(\boldsymbol{Y}_0 - \boldsymbol{\mu}^{(q)})$$

(since at convergence $\boldsymbol{Y}^{(q)} = \boldsymbol{Y}^{(q-1)} = \boldsymbol{Y}$) so that

$$\boldsymbol{Y}^{(q)} = \boldsymbol{X}(\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{Y}^{(q)} - \boldsymbol{G}^{(q)}(\boldsymbol{Y}_0 - \boldsymbol{\mu}^{(q)})$$

Hence

$$(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1})\boldsymbol{Y}^{(q)} = -\boldsymbol{G}^{(q)}(\boldsymbol{Y}_0 - \boldsymbol{\mu}^{(q)})$$

and

$$\boldsymbol{\mu}^{(q)} = \boldsymbol{Y}_0 + (\boldsymbol{G}^{(q)})^{-1}(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1})\boldsymbol{Y}^{(q)}$$

Now

$$\boldsymbol{\mu}^{(q)} = \mathrm{g}^{-1}(\boldsymbol{X}\hat{\boldsymbol{\beta}}^{(q)}) = \mathrm{g}^{-1}(\boldsymbol{X}(\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{Y}^{(q)}) \text{ so letting}$$

$$\boldsymbol{X}(\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{V}^{(q)})^{-1} = \boldsymbol{A}_0^{(q)} \text{ say, gives}$$

$$\boldsymbol{Y}_0 = \mathrm{g}^{-1}(\boldsymbol{A}_0^{(q)}\boldsymbol{Y}^{(q)}) - (\boldsymbol{G}^{(q)})^{-1}(\boldsymbol{I} - \boldsymbol{A}_0^{(q)})\boldsymbol{Y}^{(q)}$$

or

$$\boldsymbol{Y}_0 = \mathrm{g}^{-1}(\boldsymbol{A}_0^{(q)}\boldsymbol{Y}) - (\boldsymbol{G}^{(q)})^{-1}(\boldsymbol{I} - \boldsymbol{A}_0^{(q)})\boldsymbol{Y}$$

Note however that although this gives an explicit expression for $\boldsymbol{Y}_0$ in terms of $\boldsymbol{Y}$, no explicit form of $\boldsymbol{Y}$ in terms of the original observations $\boldsymbol{Y}_0$ exists; so that finding $\boldsymbol{Y}$ still requires iteration via the IGLS algorithm.

In general g_0 is a composite function. An explicit defintion of g_0 is possible via the IGLS algorithm since each iteration extends an initial identity function by one additional member of the composite. At convergence, the composite completely defines g_0, i.e., $g_0 = g_{(q)} \circ g_{(q-1)} \cdot \cdot \cdot \cdot \cdot \cdot \circ g_{(2)} \circ g_{(1)}$

References

del Pino, G., (1989). The Unifying Role of Iterative Generalized Least Squares in Statistical Algorithms, Statistical Science, 4, 394–408.

Nelder, J.A. and Wedderburn, R.W.M., (1972) Generalized Linear Models. Journal of the Royal Statistical Society, Series A, 135, 370–384.