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Monthly Disaggregation of a Quarterly Time Series and Forecasts of Its Unobservable Monthly Values

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The temporal disaggregation problem consists of deriving high frequency data from less frequent observations of a time series. This problem usually occurs when carrying out analysis of the economic situation. In this article, a direct solution is first proposed to disaggregate historical values of an aggregated time series in one step. A recursive approach is then used to estimate current disaggregated values of the series and a method is proposed to predict future disaggregated values. The procedures are derived from a statistical model that links the unobserved data with a preliminarily estimated series and with the series of aggregated values. It is assumed that the preliminary series can be estimated from data on related variables. Some results already established in the literature are employed to derive a theoretical solution that produces the Minimum Mean Squared Error Linear Estimator of the unobserved series. Mexico's GDP is used as an illustrative example.

Key words: ARIMA models; compatibility testing; minimum mean squared error; preliminary series.

1. Introduction

Several analysts have proposed methodologies to obtain high frequency data (say monthly) from less frequent observations (say quarterly) of such an important economic variable as Gross Domestic Product (GDP). Friedman (1962), one of the pioneers in this area, suggested that we should use related variables to estimate the unobserved one from observations of the others. However, his proposal was incomplete since that method does not produce an estimated series satisfying the accounting restrictions that the unobserved variable has to fulfill – the monthly GDP values, for example, must average to the observed quarterly figure. Some other works did pay attention to the accounting restrictions, but did not employ related variables. Such was the case of Lisman and Sandee (1964) and Cohen, Müller, and Padberg (1971).

The methods proposed by Chow and Lin (1971) and Denton (1971) are probably the most frequently used in practice nowadays, because they take into account both the information provided by related variables and the temporal restrictions on the unobserved series. Nevertheless, those methods consider the autocorrelation structure of the time

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series variable in a subjective way. In contrast, the solutions of Guerrero (1990) and Wei and Stram (1990) focused mainly on the use of the appropriate autocorrelation structure. Some other works dealing with the temporal disaggregation problem are those of Hillmer and Trabelsi (1987), Chen, Cholette, and Dagum (1997) and Nieto (1998). This article presents a method that: (a) uses related variables to obtain a preliminary series, (b) includes the appropriate autocorrelation structure (deduced from observed data) and (c) disaggregates the series in a statistically optimal way.

Section 2 presents a statistical model that relates the preliminary series with the unobserved one. Section 3 gives the solution to the direct temporal disaggregation problem, first as a theoretical result and then as a feasible method. The recursive disaggregation is studied in Section 4 as an extension of the results for the direct solution. From the results in Section 5 a method to forecast future values of the unobserved series is derived. Estimation of a preliminary series is given a detailed discussion in Section 6. The monthly disaggregation of Mexico's quarterly GDP is shown in Section 7. Section 8 concludes with some recommendations.

2. A Statistical Model

Let $\{Z_t\}$, for $t = 1, \ldots, mn$, be an unobserved series, where $n \geq 1$ denotes the number of whole periods (say quarters) and $m \geq 2$ is the intraperiod frequency (say months, in which case $m = 3$). Let us suppose that $\{W_t\}$ is a possibly nonstationary series of preliminary estimates of the unobserved data. Then, given $\{W_t\}$, the following relation holds:

$$Z_t = W_t + S_t, \text{ with } \{S_t\} \text{ a zero-mean unobserved stationary process.} \tag{2.1}$$

The model is complemented by the following assumptions.

Assumption 1. An Autoregressive and Moving Average (ARMA) model captures the dynamic structure of $\{S_t\}$, that is

$$\phi_S(B)S_t = \theta_S(B)e_t \tag{2.2}$$

where $\phi_S(B) = 1 - \phi_{S,1}B - \ldots - \phi_{S,p}B^p$ and $\theta_S(B) = 1 + \theta_{S,1}B + \ldots + \theta_{S,q}B^q$ are polynomials in the backshift operator B such that $BX_t = X_{t-1}$ for every variable X and every index t. Those polynomials are assumed to be prime, with the roots of $\phi_S(x) = 0$ and $\theta_S(x) = 0$ outside the unit circle, in such a way that they correspond to a stationary and invertible process. Besides, $\{e_t\}$ is a zero-mean Gaussian White Noise process with variance σ_e^2.

Assumption 2. The series $\{W_t\}$ can be represented by the following Autoregressive Integrated and Moving Average (ARIMA) model:

$$\phi_W(B)d(B)W_t = \theta_W(B)a_t \tag{2.3}$$

where $d(B)$ is a differencing operator that renders $\{d(B)W_t\}$ stationary, whereas $\phi_W(B)$ and $\theta_W(B)$ are the corresponding autoregressive (AR) and moving average (MA) polynomials whose roots are outside the unit circle. The process $\{a_t\}$ is zero-mean Gaussian White Noise with variance σ_a^2 and is uncorrelated with $\{e_t\}$.

Model (2.2) can be written equivalently as

$$S_t = \psi_S(B)e_t \tag{2.4}$$

with $\psi_S(B) = 1 + \psi_{S,1}B + \psi_{S,2}B^2 + \ldots$ the pure MA polynomial, obtained from the relation $\psi_S(B)\phi_S(B) = \theta_S(B)$ by equating coefficients of powers of B. Expression (2.4) leads us to

$$\mathbf{S} = \Psi_S \mathbf{e} \tag{2.5}$$

with $\mathbf{S} = (S_1, \ldots, S_{mn})'$ and $\mathbf{e} = (e_1, \ldots, e_{mn})'$, where the prime sign denotes transposition, and Ψ_S is an $mn \times mn$ lower triangular matrix with 1's on its main diagonal, $\psi_{S,1}$ on its first subdiagonal, $\psi_{S,2}$ on its second subdiagonal and so on. For (2.5) to be completely equivalent to (2.4) we require that $e_t = 0$ for $t \leq 0$.

On the other hand, the aggregated series $\{Y_1, \ldots, Y_n\}$ can be written as

$$Y_i = \sum_{j=1}^{m} c_j Z_{m(i-1)+j} \quad \text{for } i = 1, \ldots, n \tag{2.6}$$

where the c_j's are known constants defined by the type of aggregation. For instance, disaggregation means distribution if the Y_i are flow values, in which case $\mathbf{c}' = (c_1, \ldots, c_m) = (1, \ldots, 1)$. It also means distribution when the Y_i are indices or annualized flows, whereby the aggregation vector is $\mathbf{c}' = (1/m, \ldots, 1/m)$. Similarly, when the Y_i are stocks, disaggregation means interpolation of the values and $\mathbf{c}' = (0, \ldots, 0, 1)$ or $\mathbf{c}' = (1, 0, \ldots, 0)$. Let us now define the matrix $C = I \otimes \mathbf{c}'$ with $\otimes$ denoting the Kronecker product, $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ and $\mathbf{Z} = (Z_1, \ldots, Z_{mn})'$, whereby the whole set of restrictions (2.6) can be written as

$$\mathbf{Y} = C\mathbf{Z} \tag{2.7}$$

3. Direct Disaggregation

3.1. Optimal solution

Expressions (2.7) and (2.1), written in vector notation as $\mathbf{Z} = \mathbf{W} + \mathbf{S}$ with $\mathbf{W} = (W_1, \ldots, W_{mn})'$, allow us to use the Basic Combination Rule derived by Guerrero and Peña (2000). To that end note that $\mathrm{E}(\mathbf{Z}|\mathbf{W}) = \mathbf{W}$, so that $\mathbf{W}$ is the minimum Mean Squared Error Linear Estimator (MMSELE) of $\mathbf{Z}$ based on $\mathbf{W}$. It should be mentioned that rather than estimator we could have used the term predictor, but we reserve the word predictor for the forecasting situation of Section 5. Moreover, (2.5) implies that $\Sigma_S = \sigma_e^2 \Psi_S \Psi_S'$. Hence we get the following result that provides the theoretical solution to the direct disaggregation problem.

Proposition 1. The MMSELE of $\mathbf{Z}$, given $\mathbf{W}$ and $\mathbf{Y}$, is given by

$$\hat{\mathbf{Z}} = \mathbf{W} + \hat{A}(\mathbf{Y} - C\mathbf{W}) \tag{3.1}$$

with MSE matrix

$$\mathrm{MSE}(\hat{\mathbf{Z}}) = \sigma_e^2 (I_{mn} - \hat{A}C)\Psi_S \Psi_S' \tag{3.2}$$

where

$$\hat{A} = \Psi_S \Psi_S' C'(C \Psi_S \Psi_S' C')^{-1} \tag{3.3}$$

Remark. Because of the linearity of Relation (2.1), the MMSELE is provided only by the first and second moments. Consequently there will be little loss of generality if we assume multivariate normal distributions for **S** and **W**, in consistency with the Gaussian assumption for the processes involved in (2.2) and (2.3).

An estimate of Ψ_S can be obtained from an estimated model for the aggregated differences

$$\mathbf{D} = C\mathbf{S} = C\mathbf{Z} - C\mathbf{W} = \mathbf{Y} - C\mathbf{W} \tag{3.4}$$

That is, we assume that $\{D_i\}$ admits the ARMA model

$$\phi_D(L)D_i = \theta_D(L)\varepsilon_i, \quad \text{for } i = 1, \ldots, n \tag{3.5}$$

with $\phi_D(L) = 1 - \phi_{D1}L - \ldots - \phi_{DP}L^{\mathrm{P}}$ and $\theta_D(L) = 1 + \theta_{D1}L + \ldots + \theta_{DQ}L^Q$ polynomials in the backshift operator L acting on the aggregated variable. In (3.5) we use an ARMA model because the temporal aggregation of an ARMA process, in this case the process $\{S_t\}$, produces another ARMA process (see Engel 1984). Since data on $\{D_i\}$ are obtained from $\{Y_i\}$ and $\{W_t\}$, Model (3.5) can be built by applying standard time series techniques.

We can now use Wei and Stram's (1990) method to disaggregate Model (3.5). On the one hand, if the aggregated series is nonseasonal and has no hidden periodicity of order m, its ARMA(P, Q) model can be disaggregated into an ARMA(p, q) model. The latter model is such that $p = P$, $q = p + 1$ and its parameters can be obtained from the $P + Q$ original ones. Besides, when calculating the estimated MA parameters, an estimate of σ_e^2 is also obtained. On the other hand, when the aggregated series $\{D_i\}$ follows a seasonal ARMA model, the seasonality length is given by E/m and the seasonal AR and MA polynomials are $\Phi_D(L^{E/m}) = 1 - \Phi_1 L^{E/m} - \ldots - \Phi_{\mathrm{P}} L^{PE/m}$ and $\Theta_D(L^{E/m}) = 1 + \Theta_1 L^{E/m} + \ldots + \Theta_Q L^{QE/m}$. In order to obtain a model for the disaggregated series they proposed the following procedure. Firstly, choose the seasonal AR and MA polynomials as

$$\Phi_S(B^E) = 1 - \Phi_1 B^E - \ldots - \Phi_{\mathrm{P}} B^{PE} \text{ and } \Theta_S(B^E) = 1 + \Theta_1 B^E + \ldots + \Theta_Q B^{QE} \tag{3.6}$$

with the same parameter values as those of the model for the aggregated series. Secondly, deseasonalize both the aggregated and disaggregated series by means of the filters

$$FD_i = \Phi_D(L^{E/m})\Theta_D(L^{E/m})^{-1}D_i \text{ and } FS_t = \Phi_S(B^E)\Theta_S(B^E)^{-1}S_t \tag{3.7}$$

Then apply the procedure for nonseasonal series to obtain the model

$$\phi_S(B)FS_t = \theta_S(B)e_t \tag{3.8}$$

Thus the complete model for the disaggregated series of differences becomes

$$\phi_S(B)\Phi_S(B^E)S_t = \Theta_S(B^E)\theta_S(B)e_t \tag{3.9}$$

It should be noticed that the matrix MSE($\hat{\mathbf{Z}}$) of (3.2) will produce different variances for the disaggregated values $\{\hat{Z}_t\}$ because the elements in the diagonal of

$\Psi_S \Psi_S'$ are different, due in part to the initial conditions $e_t = 0$ imposed for $t \le 0$. An adjustment to correct for this nonstationarity situation consists in replacing those diagonal elements by the theoretical variance of the model. For instance, if the model is $(1 - \Phi B^E)S_t = (1 + \theta_1 B + \ldots + \theta_q B^q)e_t$, with $0 \le q \le E$, then its variance is given by $\mathrm{Var}(S_t) = (1 + \theta_1^2 + \ldots + \theta_q^2)\sigma_e^2/(1 - \Phi^2)$.

3.2. Validating the method

The assumption that $\{W_t\}$ is a true series of preliminary estimates of $\{Z_t\}$ can be validated empirically via a compatibility test. Since $\{Z_t\}$ is unobservable, the test should be carried out with the observed values of $C\mathbf{W}$ and $C\mathbf{Z} = \mathbf{Y}$. That is, given $\{W_t\}$ we know that

$$\mathbf{Y} - C\mathbf{W} = C\Psi_S \mathbf{e} \sim \mathrm{N}(\mathbf{0}, \sigma_e^2 C\Psi_S \Psi_S' C') \tag{3.10}$$

then a test statistic for the null hypothesis that the two sets of observed data are compatible, i.e., for H_0: $\mathrm{E}(\mathbf{Y}\,|\,\mathbf{W}) = C\mathbf{W}$, becomes

$$K = (\mathbf{Y} - C\mathbf{W})'(C\hat{\Psi}_S \hat{\Psi}_S' C')^{-1}(\mathbf{Y} - C\mathbf{W})/\hat{\sigma}_e^2 \tag{3.11}$$

whose asymptotic distribution, against which K should be compared, is Chi-square with n degrees of freedom.

At first sight it seems reasonable to check also that the model for $\{S_t\}$ provides an adequate representation for the estimated series $\{\hat{S}_t\}$. However, there is no reason for this to happen since the series $\{e_t\}$ in (2.4) is White Noise and $\{\hat{e}_t\}$ in the model $\hat{S}_t = \Psi_S(B)\hat{e}_t$ does not have to behave like that. Moreover, the elements $\hat{e}_t$ are linearly dependent since they are generated as linear combinations of the elements of $\hat{\mathbf{Z}} - \mathbf{W}$, where $\hat{\mathbf{Z}}$ satisfies the linear restrictions imposed by (2.7). This fact makes the $mn \times mn$ matrix $\mathrm{MSE}(\hat{\mathbf{Z}})$ become singular, with rank $mn - n$, and it can be verified that $C{\cdot}\mathrm{MSE}(\hat{\mathbf{Z}}) = 0$.

3.3. Comparison with other methods

The present method has some features that make it better than other methods in current use. Comparison of the new method with those of Chow and Lin (1971), Denton (1971), Hillmer and Trabelsi (1987), Guerrero (1990), Chen, Cholette, and Dagum (1997) and Nieto (1998), gives support to this statement.

Chow and Lin's (1971) method stems from the idea that $\mathbf{Z}$ can be expressed as the linear model

$$\mathbf{Z} = X\beta + \varepsilon \tag{3.12}$$

where X is a matrix of observations on variables related to Z and ε is a random vector satisfying $E(\varepsilon) = \mathbf{0}$ and $\mathrm{Cov}(\varepsilon) = V$. Chow and Lin obtained a result similar to our Proposition 1 that enables simultaneous estimation of $\mathbf{Z}$ and β, in such a way that the restriction imposed by $\mathbf{Y}$ is fulfilled. Their results are

$$\hat{\mathbf{Z}}_{Ch-L} = X\hat{\beta} + VC'(CVC')^{-1}(\mathbf{Y} - CX\hat{\beta}) \tag{3.13}$$

and

$$\hat{\beta} = [X'C'(CVC')^{-1}CX]^{-1}X'C'(CVC')^{-1}\mathbf{Y} \tag{3.14}$$

with

$$\text{MSE}(\hat{\mathbf{Z}}_{Ch-L}) = [X - VC'(CVC')^{-1}CX][X'C'(CVC')^{-1}CX]^{-1} \times \lfloor X' - X'C'(CVC')^{-1}CV \rfloor + \lfloor V - VC'(CVC')^{-1}CV \rfloor \quad (3.15)$$

By comparing these with (3.1) to (3.3), we see that $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Z}}_{Ch-L}$ are equal when $\mathbf{W} = X\hat{\beta}$ and $\sigma_e^2 \Psi_S \Psi_S' = V$, i.e., when the preliminary series comes from Model (3.12) and the matrix V is postulated correctly, in the sense that it coincides with the variance-covariance matrix of $\mathbf{S}$. The latter point is what introduces subjectivity into this method because V cannot be postulated from observed data. That is why some authors (e.g., Fernandez 1981 and Litterman 1983) have proposed some ways to postulate V. Furthermore, when V is postulated correctly, (3.15) becomes

$$\text{MSE}(\hat{\mathbf{Z}}_{Ch-L}) = \sigma_e^2(I_{mn} - AC)\Psi_S \Psi_S' + \sigma_e^2(I_{mn} - AC)X[X'C'(C\Psi_S\Psi_S'C')^{-1}CX]^{-1}X'(I_{mn} - AC)' \quad (3.16)$$

where we recognize the first element of this sum as $\text{MSE}(\hat{\mathbf{Z}})$, while the second one measures the variability associated with the discrepancy between $\hat{\mathbf{Z}}_{Ch-L}$ and $\hat{\mathbf{Z}}$. This discrepancy arises because $\hat{\mathbf{Z}}$ comes out by assuming that $\mathbf{W} = X\hat{\beta}$ is a known vector, whereas $\hat{\mathbf{Z}}_{Ch-L}$ and $\hat{\beta}$ are estimated jointly.

Denton's (1971) method arises from solving the following quadratic minimization problem:

$$min(\hat{\mathbf{Z}}_D - \mathbf{W})'P(\hat{\mathbf{Z}}_D - \mathbf{W}) \textit{ subject to } \mathbf{Y} = C\hat{\mathbf{Z}}_D \quad (3.17)$$

where $\mathbf{W}$ is a known vector and P is a constant matrix defined as $P = \Delta'\Delta$, with Δ a square penalty matrix to be specified by the analyst. A common choice for this function is

$$p(\hat{\mathbf{Z}}_D, \mathbf{W}) = \sum_{t=1}^{mn} [(1 - B)(\hat{Z}_{D,t} - W_t)]^2 \quad (3.18)$$

from which a lower triangular matrix Δ is obtained, with 1's on the main diagonal, -1's on the first subdiagonal and 0's everywhere else. Then Lagrangean minimization yields

$$\hat{\mathbf{Z}}_D = \mathbf{W} + P^{-1}C'(CP^{-1}C')^{-1}(\mathbf{Y} - C\mathbf{W}) \quad (3.19)$$

This estimator is again a special case of Expression (3.1), when $P^{-1} = \sigma_e^2 \Psi_S \Psi_S'$. Thus the matrix P^{-1} plays the role of V in the previous method and it has to be postulated on subjective grounds.

Hillmer and Trabelsi (1987) suggested a procedure based on the assumption that the vectors $\mathbf{Y}$ and $\mathbf{W}$ are linked to $\mathbf{Z}$ by way of

$$\mathbf{Y} = C\mathbf{Z} + \mathbf{u} \text{ and } \mathbf{W} = \mathbf{Z} + \mathbf{S} \quad (3.20)$$

with $\mathbf{u}$ a random vector distributed as $\text{N}(\mathbf{0}, \Sigma_u)$ and $\mathbf{S}$ another random vector, independent of $\mathbf{Z}$ and $\mathbf{u}$, distributed as $\text{N}(\mathbf{0}, \Sigma_S)$. Then a Bayesian argument led Hillmer and Trabelsi to an optimal combination of the available information. Their solution when the *a priori* distribution is noninformative and the restrictions imposed by $\mathbf{Y}$ are binding (i.e., when $\Sigma_u = 0$) yields the following expression (see Guerrero 1990 for a proof):

$$\hat{\mathbf{Z}}_{H-T} = \mathbf{W} + \Sigma_S C'(C\Sigma_S C')^{-1}(\mathbf{Y} - C\mathbf{W}) \quad (3.21)$$

This solution is identical to (3.1)–(3.3) if $\Sigma_S = \sigma_e^2 \Psi_S \Psi_S'$. Again, the specification of this matrix is what poses practical difficulties. Chen, Cholette, and Dagum (1997) realised these difficulties and proposed a feasible nonparametric solution to the problem of selecting Σ_S in two particular situations. Their method, which is based on spectral estimation and numerical techniques, is very well grounded in statistical theory, but its complexity limits its use in practical applications.

Guerrero (1990) proposed a model-based approach to the disaggregation problem. His solution can be written as

$$\hat{\mathbf{Z}}_G = \mathbf{W} + \Psi_W P \Psi_W' C'(C\Psi_W P \Psi_W' C')^{-1}(\mathbf{Y} - C\mathbf{W}) \tag{3.22}$$

where Ψ_W contains the pure MA weights of the ARIMA model for $\{W_t\}$ and P is a positive definite matrix defined implicitly by writing Equations (2.20)–(2.21) of Guerrero (1990) as

$$\mathbf{S} = \Psi_W \mathbf{v} \text{ with } E(\mathbf{v}|\mathbf{W}) = \mathbf{0} \text{ and } E(\mathbf{v}\mathbf{v}'|\mathbf{W}) = \sigma^2 P \tag{3.23}$$

so that from (2.5) it follows that

$$\sigma_e^2 \Psi_S \Psi_S' = \sigma^2 \Psi_W P \Psi_W' \tag{3.24}$$

Thus (3.22) is again equivalent to (3.1)–(3.3). The main distinction between that method and the present one concerns how to estimate (3.24), which is more theoretically sound now.

Lastly, a method suggested by Nieto (1998) is based on the assumption that the dynamic structures of $\{Z_t\}$ and $\{W_t\}$ admit the same AR model, with different variances in their respective White Noise processes. Nieto's solution is given by

$$\hat{\mathbf{Z}}_N = \mathbf{W} + \Psi_W \Psi_W' C'(C\Psi_W \Psi_W' C')^{-1}(\mathbf{Y} - C\mathbf{W}) \tag{3.25}$$

It can be seen that the dynamic structure of $\{W_t\}$, rather than that of $\{S_t\}$, is employed. This is due to the fact that Nieto's assumptions, which are different from (2.1) to (2.3), led him to derive the same AR structure for $\{S_t\}$ and $\{W_t\}$.

4. Recursive Disaggregation

The interest now lies in estimating $\mathbf{Z}^\tau = (Z_{m(\tau-1)+1}, \ldots, Z_{m\tau})'$ for $\tau = n+1, n+2, \ldots$. It is assumed that the following vectors are already given, namely $\mathbf{Z} = (\mathbf{Z}^{1\prime}, \mathbf{Z}^{2\prime}, \ldots, \mathbf{Z}^{n\prime})'$, $\mathbf{Z}^{n+1}, \ldots, \mathbf{Z}^{\tau-1}$. The unobservable vector $\mathbf{Z}$ can be estimated as indicated in the previous section and $\mathbf{Z}^{n+1}$ to $\mathbf{Z}^{\tau-1}$ as will be shown now. We shall also employ $\mathbf{W}$, $\mathbf{W}^{n+1}, \ldots, \mathbf{W}^{\tau-1}$ and $\mathbf{W}^\tau$ (defined analogously to the corresponding $\mathbf{Z}$'s) as well as the aggregated value

$$Y_\tau = \mathbf{c}'\mathbf{Z}^\tau \tag{4.1}$$

Following Guerrero and Martínez (1995), it is convenient to write (2.5) in terms of the AR and MA parameters of Model (2.2) as

$$\Phi_S \mathbf{S} = \Theta_S \mathbf{e} \tag{4.2}$$

where Φ_S is an mn-dimensional lower triangular matrix with 1's on its main diagonal, $-\phi_{S,1}$ on its first subdiagonal, up to $-\phi_{S,p}$ on its p-th subdiagonal and 0's everywhere else. Similarly Θ_S is defined in terms of the MA parameters $1, \theta_{S,1}, \ldots, \theta_{S,q}$. Expression

(4.2) leads us to use the original $p + q$ ARMA parameters. This is important when deriving the disaggregation formula in order to avoid infinite sums. An immediate extension of (4.2) yields

$$\left(\begin{array}{ccc|ccc} & \Phi_S & & 0 & \dots & 0 \\ \hline & \dots & & \dots & \dots & \dots \\ 0 & \dots & 0 & \Phi_{S,n} & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \Phi_{S,1} \end{array}\right) \left(\begin{array}{c} \mathbf{S} \\ \hline \dots \\ \mathbf{S}^{\tau-1} \\ \mathbf{S}^{\tau} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} & \Theta_S & & 0 & \dots & 0 \\ \hline & \dots & & \dots & \dots & \dots \\ 0 & & 0 & \Theta_{S,n} & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \Theta_{S,1} \end{array}\right) \left(\begin{array}{c} \mathbf{e} \\ \hline \dots \\ \mathbf{e}^{\tau-1} \\ \mathbf{e}^{\tau} \end{array}\right) \tag{4.3}$$

where the $\mathbf{e}$'s are defined in a similar fashion as the $\mathbf{Z}$'s and $\Phi_{S,i}$ and $\Theta_{S,i}$, for $i = 1, \dots, n$, are $m \times m$ matrices with $-\phi_{S,m(i-1)+j-k}$ and $\theta_{S,m(i-1)+j-k}$ in their respective (j, k) entries, for $j, k = 1, \dots, m$. Here we have $-\phi_{S,0} = \theta_{S,0} = 1$, $\phi_{S,j} = \theta_{S,j} = 0$ if $j < 0$, $\phi_{S,j} = 0$ if $j > p$ and $\theta_{S,j} = 0$ if $j > q$. Note also that $\Phi_{S,i} = \dots = \Phi_{S,n} = 0$ when $m(i-2) \leq p < m(i-1)$ and $\Theta_{S,i} = \dots = \Theta_{S,n} = 0$ when $m(i-2) \leq q < m(i-1)$. Then, if we follow the same steps that led Guerrero and Martínez (1995) to obtain their Expression (3.6), we get the following result.

Proposition 2. The MMSELE of $\mathbf{Z}^{\tau}$, given $\hat{\mathbf{Z}}^{\tau-j}$ and $\mathbf{W}^{\tau-j}$ for $j = 1, \dots, \tau - 1$; $\mathbf{e}^{\tau-h}$, for $h = 1, \dots, \tau - 1$; $\mathbf{W}^{\tau}$ and Y_{τ}, is

$$\hat{\mathbf{Z}}^{\tau} = (I_m - \hat{\mathbf{A}}^* \mathbf{c}')\left\{\mathbf{W}^{\tau} - \Phi_{S,1}^{-1}\left[\sum_{j=1}^{[p/m]} \Phi_{S,j+1}(\hat{\mathbf{Z}}^{\tau-j} - \mathbf{W}^{\tau-j}) - \sum_{h=1}^{[q/m]} \Theta_{S,h+1}\mathbf{e}^{\tau-h}\right]\right\} + \hat{\mathbf{A}}^* Y_{\tau} \tag{4.4}$$

where $[x]$ denotes the integer part function of x and the sum involved is zero when the upper limit of the sum is zero. Moreover, the MSE matrix is given by

$$\mathrm{MSE}(\hat{\mathbf{Z}}^{\tau}) = \sigma_e^2 (I_m - \hat{\mathbf{A}}^* \mathbf{c}')\Phi_{S,1}^{-1}\Theta_{S,1}\Theta_{S,1}'\Phi_{S,1}'^{-1} \tag{4.5}$$

where

$$\hat{\mathbf{A}}^* = \Phi_{S,1}^{-1}\Theta_{S,1}\Theta_{S,1}'\Phi_{S,1}'^{-1}c(c'\Phi_{S,1}^{-1}\Theta_{S,1}\Theta_{S,1}'\Phi_{S,1}'^{-1}c)^{-1} \tag{4.6}$$

The error appearing in (4.4) can be estimated as

$$\hat{\mathbf{e}}^{\tau} = \hat{\Theta}_{S,1}^{-1}\left[\sum_{j=0}^{[p/m]} \hat{\Phi}_{S,j+1}(\hat{\mathbf{Z}}^{\tau-j} - \mathbf{W}^{\tau-j}) - \sum_{h=1}^{[q/m]} \hat{\Theta}_{S,h+1}\hat{\mathbf{e}}^{\tau-h}\right] \tag{4.7}$$

where it is assumed that $\hat{\mathbf{e}}^{\tau-u} = \hat{\mathbf{e}}^{\tau-u-1} = \dots = \hat{\mathbf{e}}^{1} = \mathbf{0}$ for $u = max\{[p/m], [q/m]\} + 1$.

It should be stressed that the variability, as measured by $\mathrm{MSE}(\hat{\mathbf{Z}}^{\tau})$, stays constant for all periods to be recursively disaggregated. On the other hand, the test statistic to validate compatibility between $\mathbf{W}^\tau$ and $\mathbf{Z}^\tau$ becomes

$$K = (Y_\tau - \mathbf{c}'\overline{\mathbf{W}}^\tau)^2/(\hat{\sigma}_e^2 \mathbf{c}' \hat{\mathbf{\Phi}}_{S,1}^{-1} \hat{\Theta}_{S,1} \hat{\Theta}_{S,1}' \hat{\mathbf{\Phi}}_{S,1}'^{-1} \mathbf{c}) \tag{4.8}$$

with

$$\overline{\mathbf{W}}^\tau = \mathbf{W}^\tau - \hat{\mathbf{\Phi}}_{S,1}^{-1} \left\{ \sum_{j=1}^{[p/m]} \hat{\mathbf{\Phi}}_{S,j+1} (\hat{\mathbf{Z}}^{\tau-j} - \mathbf{W}^{\tau-j}) - \sum_{h=1}^{[q/m]} \hat{\Theta}_{S,h+1} \hat{\mathbf{e}}^{\tau-h} \right\} \tag{4.9}$$

This statistic should be compared against a Chi-square distribution with 1 degree of freedom.

The recursive approach is useful to estimate the values of each complete period as soon as another aggregated value Y_i is observed. The fact that the previously disaggregated figures remain unaffected by the recursive disaggregation results and its computational efficiency, makes the recursive method a good complement to, rather than a competitor of, the direct method. The recursion is appropriate to estimate current figures of the series, whereas the direct method is adequate to estimate a historical record.

5. Forecasting Future Disaggregated Values

Once the vectors $\hat{\mathbf{W}}_N = (\hat{W}_1, \ldots, \hat{W}_{mN})'$ and $\hat{\mathbf{S}}_N = (\hat{S}_1, \ldots, \hat{S}_{mN})'$ have been estimated, $\hat{\mathbf{W}}_N$ as indicated in Section 6 and $\hat{\mathbf{S}}_N$ as in Sections 3 and 4, we relate them as in (2.1) to obtain $\hat{Z}_t = \hat{W}_t + \hat{S}_t$ for $t = 1, \ldots, mN$, where $mN \geq mn$ is the number of previously disaggregated values.

The problem now is to forecast the vector $\mathbf{Z}_F = (Z_{mN+1}, \ldots, Z_{mN+H})'$, with $H \geq 1$ the forecast horizon, when there are no aggregate values $\{Y_i\}$ available for $i > N$. As in Nieto (1998), we consider two situations concerning the availability of preliminary values: (1) No preliminary observations exist for $t > mN$; and (2) $W_{mN+1}, \ldots, W_{mN+\eta}$ are available for $1 \leq \eta \leq H$.

In the first case the forecast is defined as $\hat{\mathbf{Z}}_F^{(1)} = (\hat{Z}_{mN}^{(1)}(1), \ldots, \hat{Z}_{mN}^{(1)}(H))'$ with

$$\hat{Z}_{mN}^{(1)}(h) = E(\hat{W}_{mN+h}|\hat{\mathbf{W}}_N) + E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) \quad \text{for } h = 1, \ldots, H \tag{5.1}$$

where $E(\hat{W}_{mN+h}|\hat{\mathbf{W}}_N)$ and $E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N)$ are obtained from their respective Models (2.3) and (2.2). Thus the forecasts satisfy

$$\phi_W(B)d(B)E(\hat{W}_{mN+h}|\hat{\mathbf{W}}_N) = \theta_W(B)\hat{a}_{mN+h} \quad \text{and} \quad \phi_S(B)E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) = \theta_S(B)\hat{e}_{mN+h} \tag{5.2}$$

with $E(\hat{W}_{mN+h-j}|\hat{\mathbf{W}}_N) = \hat{W}_{mN+h-j}$ and $E(\hat{S}_{mN+h-j}|\hat{\mathbf{S}}_N) = \hat{S}_{mN+h-j}$ if $j \geq h$, and $\hat{a}_{mN+h} = \hat{e}_{mN+h} = 0$ for $h \geq 1$. Now we have

$$\begin{aligned} \hat{W}_{mN+h} - E(\hat{W}_{mN+h}|\hat{\mathbf{W}}_N) &= \sum_{j=0}^{h-1} \psi_{W,j}\hat{a}_{mN+h-j} \text{ and} \\ S_{mN+h} - E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) &= \sum_{j=0}^{h-1} \psi_{S,j}\hat{e}_{mN+h-j} \end{aligned} \tag{5.3}$$

with the ψ_W weights coming from $\psi_W(B)\phi_W(B)d(B) = \theta_W(B)$ and the ψ_S's from (2.5). Expressions (5.3) can be written as

$$\hat{\mathbf{W}}_F - E(\hat{\mathbf{W}}_F|\mathbf{W}_N) = \Psi_W^{(H)}\hat{\mathbf{a}}_F \quad \text{and} \quad \hat{\mathbf{S}}_F - E(\hat{\mathbf{S}}_F|\hat{\mathbf{S}}_N) = \Psi_S^{(H)}\hat{\mathbf{e}}_F \tag{5.4}$$

where the vectors with subindex F are defined analogously to $\mathbf{Z}_F$, and $\Psi_W^{(H)}$ is a lower triangular matrix with elements $1, \psi_{W,1}, \ldots, \psi_{W,H-1}$ on its first column, 0, 1, $\psi_{W,1}, \ldots, \psi_{W,H-2}$ on its second column and so on. $\Psi_S^{(H)}$ is defined in the same way as $\Psi_W^{(H)}$. From (5.4) it follows that

$$\hat{\mathbf{Z}}_F - \hat{\mathbf{Z}}_F^{(1)} = \mathbf{W}_F + \mathbf{S}_F - E(\hat{\mathbf{W}}_F|\hat{\mathbf{W}}_N) - E(\hat{\mathbf{S}}_F|\hat{\mathbf{S}}_N) = \Psi_W^{(H)}\hat{\mathbf{a}}_F + \Psi_S^{(H)}\hat{\mathbf{e}}_F \tag{5.5}$$

Then, since $\{a_t\}$ and $\{e_t\}$ are uncorrelated, we get

$$\mathrm{MSE}(\hat{\mathbf{Z}}_F^{(1)}) = \mathrm{Cov}(\Psi_W^{(H)}\hat{\mathbf{a}}_F + \Psi_S^{(H)}\hat{\mathbf{e}}_F) = \sigma_a^2\Psi_W^{(H)}\Psi_W^{(H)\prime} + \sigma_e^2\Psi_S^{(H)}\Psi_S^{(H)\prime} \tag{5.6}$$

In Case (2) we have

$$\hat{Z}_{mN+h}^{(2)} = \begin{cases} \hat{W}_{mN+h} + E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) & \text{if } h = 1, \ldots, \eta \\ E(\hat{W}_{mN+h}|\hat{\mathbf{W}}_N) + E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) & \text{if } h = \eta+1, \ldots, H \end{cases} \tag{5.7}$$

with $E(\hat{\mathrm{W}}_{mN+h-j}|\hat{\mathbf{W}}_N) = \hat{W}_{mN+h-j}$ if $j \geq h-\eta$ and $E(\hat{S}_{mN+h-j}|\hat{\mathbf{S}}_N) = S_{mN+h-j}$ if $j \geq h$. Therefore

$$\hat{Z}_{mN+h} - \hat{Z}_{mN+h}^{(2)} = \begin{cases} \hat{S}_{mN+h} - E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) & \text{if } h = 1, \ldots, \eta \\ \hat{W}_{mN+h} - E(\hat{W}_{mN+h}|\hat{\mathbf{W}}_N) + \hat{S}_{mN+h} - E(\hat{S}_{mN+h}|\hat{\mathbf{S}}_N) & \text{if } h = \eta+1, \ldots, N \end{cases} \tag{5.8}$$

so that

$$\hat{\mathbf{Z}}_F - \hat{\mathbf{Z}}_F^{(2)} = \begin{pmatrix} \hat{\mathbf{e}}_{mN+1} \\ \ldots \\ \sum_{j=0}^{\eta-1}\psi_{S,j}\hat{\mathbf{e}}_{mN+\eta-j} \\ \hat{a}_{mN+\eta+1} + \sum_{j=0}^{\eta}\psi_{S,j}\hat{e}_{mN+\eta+1-j} \\ \ldots \\ \sum_{j=0}^{H-\eta-1}\psi_{W,j}\hat{a}_{mN+H-j} + \sum_{j=0}^{H-1}\psi_{S,j}\hat{e}_{mN+H-j} \end{pmatrix}$$

$$= \Psi_S^{(H)}\hat{\mathbf{e}}_F + \begin{pmatrix} 0_\eta & 0 \\ 0 & \Psi_W^{(H-\eta)} \end{pmatrix}\hat{\mathbf{a}}_F \tag{5.9}$$

with 0_η the $\eta \times \eta$ zero matrix. Hence, the MSE matrix of the forecast vector is

$$\mathrm{MSE}(\hat{\mathbf{Z}}_F^{(2)}) = \sigma_e^2\Psi_S^{(H)}\Psi_S^{(H)\prime} + \sigma_a^2\begin{pmatrix} 0_\eta & 0 \\ 0 & \Psi_W^{(H-\eta)}\Psi_W^{(H-\eta)\prime} \end{pmatrix} \tag{5.10}$$

In summary, the forecast of $\hat{\mathbf{Z}}_F$ is the sum of the forecasts of $\hat{\mathbf{W}}_F$ and $\hat{\mathbf{S}}_F$ obtained separately. Its MSE matrix is the sum of the corresponding matrices for $E(\hat{\mathbf{W}}_F|\hat{\mathbf{W}}_N)$ and $E(\hat{\mathbf{S}}_F\,|\,\hat{\mathbf{S}}_N)$.

6. Estimating the Preliminary Series

In practical applications the preliminary series has to be estimated from variables related to Z. Friedman's (1962) original idea is still in use, as evidenced by the works of Braun (1990), Abeysinghe and Lee (1998) and Nieto (1998), among others. The variables related to Z will be denoted by $X_1, \ldots, X_G$ where $G \geq 1$. These are chosen because their intraperiod (say monthly) movements are deemed to be correlated with those of $\{Z_t\}$. Stemming from this belief the following equation arises:

$$W_t = \beta_1 X_{1t} + \ldots + \beta_G X_{Gt}, \quad \text{for } t = 1, \ldots, mn \tag{6.1}$$

where the coefficients $\beta_1, \ldots, \beta_G$ should be estimated from the data. Thus we postulate the multiple linear regression model

$$Z_t = \beta_1 X_{1t} + \ldots + \beta_G X_{Gt} + \varepsilon_t, \quad \text{for } t = 1, \ldots, mn \tag{6.2}$$

from which a model for the aggregated variable is obtained, that is

$$Y_i = \beta_1 X_{1i}^a + \ldots + \beta_G X_{Gi}^a + \varepsilon_i^a, \quad \text{for } i = 1, \ldots, n \tag{6.3}$$

where $X_1^a, \ldots, X_G^a$ and ε^a are linked to $X_1, \ldots, X_G$ and ε, as Y, is linked to Z. Thus we have

$$X_{gi}^a = \sum_{j=1}^{m} c_j X_{g,m(i-1)+j} \quad \text{for } g = 1, \ldots, G \tag{6.4}$$

and the same thing happens with ε_i^a as a function of ε_t. The parameters β can be estimated from (6.3) by ordinary least squares and the estimated coefficients can be plugged in (6.1) to estimate the preliminary series.

In order to get a reasonable preliminary estimate, we suggest the following criteria to choose a variable X_g: (i) that it admits an adequate economic interpretation in its relation to Z; (ii) that it satisfies Friedman's (1962) assumption of high intraperiod correlation with Z; (iii) that the length of the series $\{X_{gt}\}$ be long enough to cover data from $t = 1, \ldots, mn$ and that it be observed also for $t > mn$; (iv) that it be observed timely, for the recursive disaggregation to make sense; and (v) that its measurement method does not change with time. From (2.1) it must be clear that there is no allowance for structural changes or outliers in $\{S_t\}$, hence not in $\{D_i\}$ either. This fact implies that the preliminary series must reflect all the effects that the true series $\{Z_t\}$ is expected to have, including those associated with structural breaks and outliers. Therefore when subject matter knowledge of the variable Z indicates that such an effect must be present, we should consider its presence also in W through an appropriate indicator variable X_g.

7. Monthly Disaggregation of Mexico's GDP

In Mexico, as in many other countries, GDP is measured only on a quarterly basis. If monthly data were available, the basic need of analyzing the economic situation could be satisfied more reasonably. A step in this direction was given by INEGI (Instituto Nacional de Estadística, Geografía e Informática, Mexico) when it started to produce figures of a monthly index of global economic activity (IMGAE, Base 1993 = 100) in January 1993. We thus have access to Y = GDP (quarterly) and X = IMGAE (monthly). IMGAE is a general indicator that only takes into account the industrial sector and the services sector of the Mexican economy, but its high intraperiod correlation with GDP will be evident below. Besides being calculated with the same methodology as GDP, although with less coverage, it satisfies all the criteria mentioned in the previous section so that it can be considered a reasonable indicator on a priori grounds. When this work was carried out, the available data on IMGAE ran from January 1993 through December 1999. These data appear in Table 2 of the Appendix, while Table 1 contains the quarterly GDP.

A quarterly indicator, INDIAGR, was built by averaging the monthly figures of IMGAE and a linear regression model was fitted to the aggregated data, yielding the following results for quarters $i = 1, \ldots, 28$ (that is, 1993: I to 1999: IV) with standard errors in parentheses

$$\underset{}{\mathrm{GDP}_i} = \underset{(20231.38)}{20311.79} + \underset{(188.04)}{12359.80}\mathrm{INDIAGR}_i, \quad \overline{R}^2 = 0.9938, \quad DW = 2.23 \tag{7.1}$$

Then the monthly preliminary data were obtained for $t = 1, \ldots 84$ (January 1993 to December 1999) with the equation

$$W_t = 20311.79 + 12359.80\mathrm{IMGAE}_t \tag{7.2}$$

These figures were aggregated to the quarter to get the values of GDPAGR_i for $i = 1, \ldots, 28$, as well as the differences $D_i = \mathrm{GDP}_i - \mathrm{GDPAGR}_i$, also shown in Table 1.

Equation (7.1) shows a strong linear relationship between GDP and INDIAGR, as measured by $\overline{R}^2$, and the Durbin–Watson statistic does not show evidence of inadequacy. Since the estimated slope is more than 65 times its standard error, INDIAGR was considered a good predictor of GDP. Even though the intercept is not significantly different from zero, it was included in Equation (7.2) to avoid possible biases when predicting $\{W_t\}$. The autocorrelations for series $\{D_i\}$ were calculated from the 28 data points of the series and they allowed us to identify a seasonal ARMA model for $\{D_i\}$. The estimation results of such a model are

$$(1 - \underset{(0.1730)}{0.6001}L^4)D_i = \hat{\varepsilon}_i \quad \text{with } \hat{\sigma}_\varepsilon = 6905.45 \tag{7.3}$$

The Ljung-Box statistic became $Q'(5) = 4.95$. When comparing this value against a Chi-square distribution with 2 degrees of freedom, there is no reason to doubt the model's adequacy. In order to get a disaggregated model, the following seasonal AR polynomial was defined:

$$\hat{\Phi}(B) = 1 - 0.6001B^{12} \tag{7.4}$$

and a deseasonalized series was obtained from $\{D_i\}$ by applying the filter

$$FD_i = D_i - 0.6001D_{i-4} \quad \text{for } i = 5, \ldots, 28. \tag{7.5}$$

Thus the nonseasonal AR and MA polynomials of the disaggregated series of differences were identified by analyzing the sample autocorrelations of $\{FD_i\}$. None of these autocorrelations differed significantly from zero. Hence the polynomial orders were chosen as $p = 0$ and $q = p + 1 = 1$, following Wei and Stram's (1990) method.

Since the aggregation involved in the present case is of the form

$$FD_i = \tfrac{1}{3}(1 + B + B^2)FS_{3i} \tag{7.6}$$

then the autocovariances of the aggregated and disaggregated series are

$$\begin{aligned}\gamma_{FD}(0) &= \tfrac{1}{9}(1+B+B^2)^2\gamma_{FS}(2)\\ &= \tfrac{1}{9}[\gamma_{FS}(-2)+2\gamma_{FS}(-1)+3\gamma_{FS}(0)+2\gamma_{FS}(1)+\gamma_{FS}(2)]\end{aligned} \tag{7.7a}$$

and

$$\begin{aligned}\gamma_{FD}(1) &= \tfrac{1}{9}(1+B+B^2)^2\gamma_{FS}(5)\\ &= \tfrac{1}{9}[\gamma_{FS}(1)+2\gamma_{FS}(2)+3\gamma_{FS}(3)+2\gamma_{FS}(4)+\gamma_{FS}(5)]\end{aligned} \tag{7.7b}$$

where it is assumed that $\gamma_{FD}(k) = 0 = \gamma_{FS}(k)$ for $k \neq 0, \pm 1$. Thus the following system of equations was obtained

$$\begin{pmatrix}\gamma_{FD}(0)\\ \gamma_{FD}(1)\end{pmatrix} = \begin{pmatrix}3/9 & 4/9\\ 0 & 1/9\end{pmatrix}\begin{pmatrix}\gamma_{FS}(0)\\ \gamma_{FS}(1)\end{pmatrix} \tag{7.8}$$

which, given that $\hat{\gamma}_{FD}(0) = 47647902.75$ and $\hat{\gamma}_{FD}(1) = \hat{\gamma}_{FD}(0)\hat{\rho}_{FD}(1) = 8187991.91$, yields

$$\begin{pmatrix}\hat{\gamma}_{FS}(0)\\ \hat{\gamma}_{FS}(1)\end{pmatrix} = \begin{pmatrix}3 & -12\\ 0 & 9\end{pmatrix}\begin{pmatrix}47647902.75\\ 8187991.91\end{pmatrix} = \begin{pmatrix}44687805.38\\ 73691927.91\end{pmatrix} \tag{7.9}$$

This result is inadmissible because it leads one to estimate the first autocorrelation of $\{FS_t\}$ as $\hat{\rho}_{FS}(1) = \hat{\gamma}_{FS}(1)/\hat{\gamma}_{FS}(0) = 1.6490$, which does not make any sense, since the absolute value of the first autocorrelation of an MA(1) model must not be larger than 0.5.

A possible explanation of the aforementioned result is that some hidden periodicity of order $m = 3$ is present in $\{FS_t\}$. Thus the MA polynomial was assumed of order $q = 3$, so that $\gamma_{FD}(k) = 0$ for $k \neq 0, \pm 1$ and $\gamma_{FS}(k) = 0$ for $k \neq 0, \pm 3$. The corresponding system of equations became

$$\begin{pmatrix}\gamma_{FD}(0)\\ \gamma_{FD}(1)\end{pmatrix} = \begin{pmatrix}3/9 & 0\\ 0 & 3/9\end{pmatrix}\begin{pmatrix}\gamma_{FS}(0)\\ \gamma_{FS}(3)\end{pmatrix} \tag{7.10}$$

with solution

$$\begin{pmatrix}\hat{\gamma}_{FS}(0)\\ \hat{\gamma}_{FS}(3)\end{pmatrix} = \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}47647902.75\\ 8187991.91\end{pmatrix} = \begin{pmatrix}142943708.30\\ 24563975.72\end{pmatrix} \tag{7.11}$$

These autocovariances enabled us to estimate the MA(3) parameter of $FS_t = (1+\theta_3 B^3)e_t$. Since the theoretical autocovariances for that model are $\gamma_{FS}(0) = (1+\theta_3^2)\sigma_e^2$ and $\gamma_{FS}(3) = \theta_3\sigma_e^2$ the estimator $\hat{\theta}_3$ came out by solving the equation

$$\hat{\gamma}_{FS}(3) - \hat{\gamma}_{FS}(0)\hat{\theta}_3 + \hat{\gamma}_{FS}(3)\hat{\theta}_3^2 = 0 \tag{7.12}$$

That is,

$$\hat{\theta}_3 = \frac{\hat{\gamma}_{FS}(0) \pm \sqrt{\hat{\gamma}_{FS}^2(0) - 4\hat{\gamma}_{FS}(3)}}{2\hat{\gamma}_{FS}(3)} = \frac{\hat{\gamma}_{FS}(0)}{2\hat{\gamma}_{FS}(3)} \pm \sqrt{\{\hat{\gamma}_{FS}(0)/2\hat{\gamma}_{FS}(3)\}^2 - 1} \tag{7.13}$$

By plugging the estimated values in this equation we obtained $\hat{\theta}_{31} = 0.1772$ and $\hat{\theta}_{32} = 5.6420$. Then $\hat{\theta}_3$ was chosen as the former value, in order to ensure invertibility of the model. Hence the estimated model for series $\{S_t\}$ is given by

$$(1-0.6001B^{12})S_t = (1+0.1772B^3)\hat{e}_t \quad \text{with} \quad \hat{\sigma}_e^2 = \hat{\gamma}_{FS}(3)/\hat{\theta}_3 = 138589937.5 \tag{7.14}$$

Such a model has the form $(1 - \Phi B^E)S_t = (1 + \theta B^3)e_t$, hence the weights for its pure MA representation are obtained as

$$\psi_{3+12(j-1)} = \Phi^{j-1}\theta \text{ for } j = 1,2,\ldots, \psi_{12j} = \Phi^j \text{ for } j = 0,1,\ldots, \text{ and } \psi_j = 0 \text{ otherwise.} \tag{7.15}$$

To correct for nonconstant variance, the diagonal elements of $\Psi_S \Psi_S'$ were modified so that they take on the value of the variance $\mathrm{Var}(S_t)/\sigma_e^2 = (1+\theta^2)/(1-\Phi^2)$. Lastly, it can be verified that the matrices required by the recursive disaggregation become

$$\Phi_{S,1} = I_3, \quad \Phi_{S,2} = \Phi_{S,3} = \Phi_{S,4} = 0_{3\times3}, \quad \Phi_{S,5} = -\Phi I_3 \quad \text{and} \quad \Theta_{S,1} = I_3, \quad \Theta_{S,2} = -\theta I_3 \tag{7.16}$$

Once the estimated model for series $\{S_t\}$ is available, Proposition 1 can be applied to disaggregate GDP directly. The corresponding results of this application are shown in Table 2 and Figure 1. The figure shows that the preliminary and the direct disaggregated series follow each other very closely, even in such periods as that of year 1995, when the Mexican crisis took place. Standard errors of the estimates in the table serve to calculate the 95% prediction band that has constant amplitude. The standard errors are given by $\hat{se}_t = \sqrt{\hat{\mathrm{Var}}(S_t)}$ and the 95% limits were calculated on the normality assumption for S_t, by means of

$$\hat{Z}_t \pm 1.96\,\hat{se}_t \tag{7.17}$$

Table 2 also presents the annual rates of growth of the disaggregated series (in percentages), which can be compared with those of IMGAE. The behavior of these two series of rates is again very similar, as it should be. Besides, the direct disaggregation was validated by the compatibility statistic whose value $K_{calc} = 25.90$ (with 28 degrees of

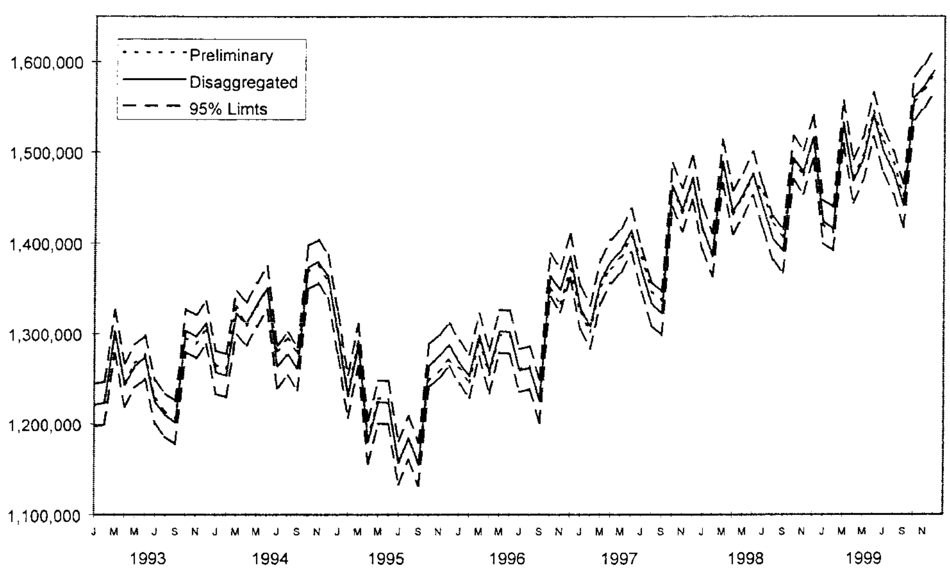

Fig. 1. Monthly disaggregation of Mexico's Real GDP (in millions of pesos at 1993 prices). Preliminary series and direct disaggregated series, with 95% limits

freedom) yielded a significance level of 0.58. This result lent empirical support to the assumption of compatibility between preliminary and disaggregated series.

In this case the direct disaggregation suffices to obtain the disaggregated figures of GDP for the historical record of the series (1993–1999). Therefore the recursive procedure was not applied at this stage. We then proceeded to forecast Mexico's GDP. To that end, an ARIMA model for the preliminary series $\{W_t\}$ was built, yielding the following results:

$$(1-B)(1-B^{12})W_t = (1 - \underset{(0.1241)}{0.3438}B^{10})(1 - \underset{(0.0895)}{0.8684}B^{12})\hat{a}_t, \quad \hat{\sigma}_a = 23462.34 \tag{7.18}$$

where the Ljung-Box statistic $Q' = 19.93$, with 15 degrees of freedom, did not provide evidence of inadequacy. We did look for an interpretation of this model, in particular of the fact that a lag 10 appears in the regular MA part, because it will be employed only as a forecasting tool.

Table 3 contains the GDP forecasts (that involve forecasts of the preliminary series), together with their standard errors and annual rate of growth, when no preliminary observations are available for the forecast horizon (January through December 2000). The estimated model (7.14) for series $\{S_t\}$ was used and 90% prediction limits were calculated with standard errors given by the squared root of the elements on the diagonal of matrix (5.6). Another set of forecasts was obtained when two preliminary observations were available (January 2000: 1,516,028.82; and February 2000: 1,536,908.89). These forecasts are shown both in Table 4 and in Figure 2 (together with their prediction limits). Figure 2 provides a visual summary of the results produced by the disaggregation and forecasting methods. We should bear in mind that the forecasts of the disaggregated series have standard errors that take into account the uncertainty in the forecasts of the preliminary series as well as that of the unobservable series of differences. Therefore the standard errors of the forecasts shown in Table 3 are larger than those for the preliminary series. The present

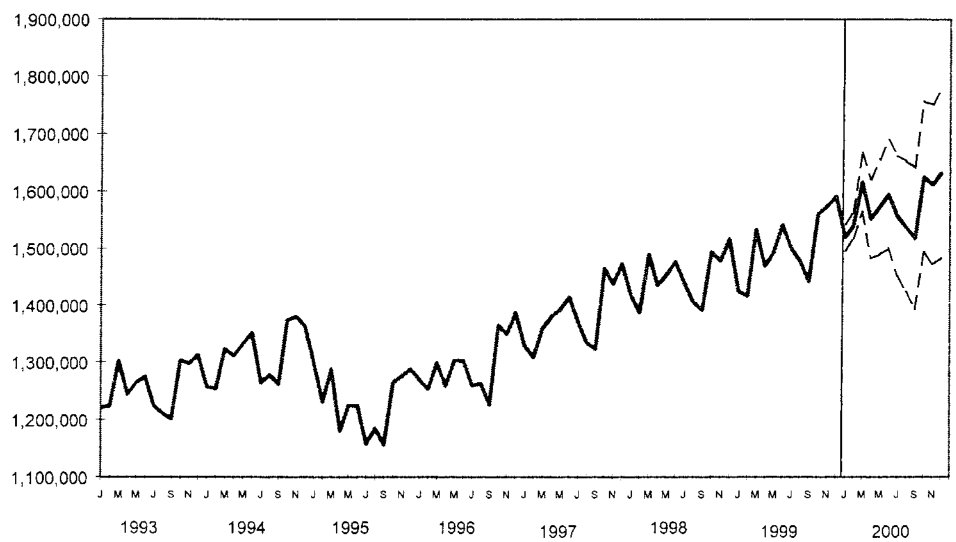

Fig. 2. Monthly disaggregation and forecasts of Mexico's Real GDP. Disaggregated series (1993–1999) and forecasts with two preliminary observations

methodology also enables us to obtain forecasts for the quarterly and yearly averages of the monthly GDP. Thus, as a complement of the monthly forecasts we also obtained the quarterly and yearly forecasts shown in Tables 3 and 4. In particular, the standard errors shown in those tables allow us to see clearly the gain in precision by having included the preliminary observations.

Finally, to illustrate the use of the recursive procedure let us consider now the situation where an additional whole period observation (GDP of the first quarter of 2000: 1,567,276.75) is available. Of course, the monthly values of IMGAE are also available, hence also the preliminary estimates of the monthly GDP, calculated as in (7.2). In this case the compatibility statistic yielded the value $K_{calc} = 2.03$, which, when compared against a Chi-square distribution with 1 degree of freedom, provided a significance level of 0.15, so that there was no reason to doubt the compatibility between preliminary data and aggregated observations. The resulting estimated monthly values of GDP for the first quarter of year 2000 appear in Table 5. There we can observe essentially the same behavior as that of the estimates reported in Table 2.

8. Conclusions and Recommendations

The proposed procedures are supported by several intermediate and already known results that are optimal when it comes to solving a specific part of the problem of temporal disaggregation and forecasting of an unobservable time series. Each of those results is derived on the basis of assumptions that must be empirically validated in order to maintain its optimality. The most important assumptions were mentioned when deriving the theoretical results here employed. However, another assumption that must be taken into account is that the series of differences $\{S_t\}$ is unaffected by structural breaks. Besides, it should be clear that the ARMA model for the aggregated differences $\{D_i\}$ is obtained with possibly a very small amount of data since only n whole periods are available. Thus this model may change its structure gradually, but no sudden changes are allowed.

The statistical quality of the related variables employed by the preliminary series should also be monitored, trying to detect possible inadequacies. Whenever possible they should be improved to cover more economic sectors and geographic regions. Finally, it should be mentioned that the application here detailed for disaggregating a Real GDP series may be carried out also with a Current GDP series. Such an application would basically imply working first with the GDP implicit deflator to disaggregate it and then deriving the disaggregated Current GDP from the disaggregated figures of Real GDP and its implicit deflator. Many other applications can be devised for this methodology when analyzing national accounts, including simultaneous disaggregation of multiple time series.

Appendix

Table 1. Mexico's Real GDP (in millions of pesos at 1993 prices), aggregated preliminary series and estimated series of differences

	Quarter	GDP	INDIAGR	GDPAGR	D
1993	I	1,248,725.34	99.37	1,248,463.50	261.84
	II	1,260,351.97	100.60	1,263,707.25	−3,355.27
	III	1,211,579.72	96.73	1,215,916.04	−4,336.32
	IV	1,304,126.86	103.27	1,296,666.70	7,460.15
1994	I	1,277,838.03	102.37	1,285,542.89	−7,704.85
	II	1,331,435.05	105.93	1,329,626.16	1,808.89
	III	1,267,386.31	102.30	1,284,718.90	−17,332.59
	IV	1,372,142.33	109.00	1,367,529.53	4,612.80
1995	I	1,272,241.55	101.62	1,276,321.16	−4,079.61
	II	1,209,052.70	96.51	1,213,147.06	−4,094.36
	III	1,165,580.18	92.62	1,165,076.38	503.80
	IV	1,275,557.48	100.24	1,259,282.00	16,275.49
1996	I	1,273,078.05	100.81	1,266,356.12	6,721.92
	II	1,287,401.28	102.22	1,283,766.66	3,634.61
	III	1,248,655.10	99.53	1,250,431.75	−1,766.65
	IV	1,366,292.01	107.71	1,351,527.45	14,764.56
1997	I	1,331,526.94	105.64	1,325,992.75	5,534.19
	II	1,395,247.46	110.66	1,388,018.61	7,228.85
	III	1,342,047.95	108.06	1,355,948.87	−13,900.92
	IV	1,457,278.33	115.98	1,453,831.52	3,446.81
1998	I	1,430,820.67	114.01	1,429,452.38	1,368.29
	II	1,454,490.59	115.82	1,451,850.95	2,639.63
	III	1,411,536.62	113.81	1,426,933,67	−15,397.06
	IV	1,495,691.40	119.11	1,492,464.42	3,226.98
1999	I	1,457,161,35	115.95	1,453,470.11	3,691.23
	II	1,500,167.45	120.05	1,504,081.03	−3,913.58
	III	1,472,607.44	118.39	1,483,604.18	−10,996.74
	IV	1,574,096.55	125.41	1,570,398.65	3,697.90

Table 2. Results of the direct disaggregation of Mexico's Real GDP (1993–1999)

Year	IMGAE	Annual rate	Preliminary series	Disaggregated series	Standard error	Annual rate
1993	97.10	–	1,220,447.96	1,220,709.80	12,203.63	–
	97.30	–	1,222,919.92	1,223,181.76	12,203.63	–
	103.70	–	1,302,022.61	1,302,284.45	12,203.63	–
	99.20	–	1,246,403.53	1,243,048.26	12,203.63	–
	100.90	–	1,267,415.19	1,264,059.91	12,203.63	–
	101.70	–	1,277,303.02	1,273,947.75	12,203.63	–
	97.70	–	1,227,863.84	1,223,527.52	12,203.63	–
	96.60	–	1,214,268.06	1,209,931.74	12,203.63	–
	95.90	–	1,205,616.21	1,201,279.89	12,203.63	–
	103.20	–	1,295,842.72	1,303,302.87	12,203.63	–
	102.70	–	1,289,662.82	1,297,122.97	12,203.63	–
	103.90	–	1,304,494.57	1,311,954.73	12,203.63	–
1994	100.70	3.71	1,264,943.23	1,257,238.37	12,203.63	2.99
	100.40	3.19	1,261,235.29	1,253,530.43	12,203.63	2.48
	106.00	2.22	1,330,450.14	1,322,745.29	12,203.63	1.57
	104.30	5.14	1,309,438.49	1,311,247.39	12,203.63	5.49
	106.00	5.05	1,330,450.14	1,332,259.04	12,203.63	5.40
	107.50	5.70	1,348,989.84	1,350,798.73	12,203.63	6.03
	102.00	4.40	1,281,010.96	1,263,678.37	12,203.63	3.28
	103.10	6.73	1,294,606.74	1,277,274.14	12,203.63	5.57
	101.80	6.15	1,278,539.00	1,261,206.41	12,203.63	4.99
	109.10	5.72	1,368,765.51	1,373,378.31	12,203.63	5.38
	109.60	6.72	1,374,945.41	1,379,558.21	12,203.63	6.36
	108.30	4.23	1,358,877.67	1,363,490.47	12,203.63	3.93
1995	103.76	3.04	1,302,774.88	1,298,695.27	12,203.63	3.30
	98.24	−2.15	1,234,494.18	1,230,414.57	12,203.63	−1.84
	102.86	−2.96	1,291,694.42	1,287,614.81	12,203.63	−2.66
	94.13	−9.75	1,183,740.18	1,179,645.82	12,203.63	−10.04
	97.72	−7.81	1,228,164.38	1,224,070.02	12,203.63	−8.12
	97.67	−9.14	1,227,536.63	1,223,442.26	12,203.63	−9.43
	91.93	−9.87	1,156,573.18	1,157,076.98	12,203.63	−8.44
	94.14	−8.70	1,183,805.50	1,184,309.31	12,203.63	−7.28
	91.79	−9.83	1,154,850.46	1,155,354.26	12,203.63	−8.39
	99.33	−8.96	1,247,976.30	1,264,251.79	12,203.63	−7.95
	100.20	−8.58	1,258,714.58	1,274,990.07	12,203.63	−7.58
	101.20	−6.55	1,271,155.11	1,287,430.60	12,203.63	−5.58
1996	100.41	−3.23	1,261,385.11	1,268,107.04	12,203.63	−2.36
	99.17	0.95	1,245,979.55	1,252,701.48	12,203.63	1.81
	102.87	0.00	1,291,703.70	1,298,425.63	12,203.63	0.84
	99.83	6.05	1,254,174.01	1,257,808.63	12,203.63	6.63
	103.46	5.87	1,299,047.68	1,302,682.30	12,203.63	6.42
	103.38	5.84	1,298,078.29	1,301,712.91	12,203.63	6.40
	100.36	9.17	1,260,800.10	1,259,033.45	12,203.63	8.81
	100.61	6.88	1,263,800.36	1,262,033.71	12,203.63	6.56
	97.61	6.33	1,226,694.79	1,224,928.13	12,203.63	6.02
	107.55	8.28	1,349,608.33	1,364,372.89	12,203.63	7.92
	106.26	6.05	1,333,678.56	1,348,443.12	12,203.63	5.76
	109.30	8.01	1,371,295.45	1,386,060.01	12,203.63	7.66

Table 2. Continued

Year	IMGAE	Annual rate	Preliminary series	Disaggregated series	Standard error	Annual rate
1997	105.41	4.98	1,323,136.78	1,328,670.97	12,203.63	4.78
	103.71	4.58	1,302,174.16	1,307,708.34	12,203.63	4.39
	107.80	4.80	1,352,667.32	1,358,201.51	12,203.63	4.60
	109.41	9.60	1,372,628.54	1,379,857.39	12,203.63	9.70
	110.38	6.69	1,384,574.20	1,391,803.05	12,203.63	6.84
	112.18	8.51	1,406,853.10	1,414,081.95	12,203.63	8.63
	110.35	9.95	1,384,238.33	1,370,337.41	12,203.63	8.84
	107.35	6.70	1,347,103.58	1,333,202.67	12,203.63	5.64
	106.49	9.10	1,336,504.69	1,322,603.77	12,203.63	7.97
	116.50	8.32	1,460,242.84	1,463,689.66	12,203.63	7.28
	114.30	7.57	1,433,064.70	1,436,511.52	12,203.63	6.53
	117.14	7.17	1,468,187.02	1,471,633.83	12,203.63	6.17
1998	112.82	7.03	1,414,767.90	1,416,136.20	12,203.63	6.58
	110.46	6.51	1,385,581.66	1,386,949.95	12,203.63	6.06
	118.75	10.16	1,488,007.58	1,489,375.87	12,203.63	9.66
	114.19	4.36	1,431,623.19	1,434,262.82	12,203.63	3.94
	115.70	4.82	1,450,325.27	1,452,964.90	12,203.63	4.39
	117.58	4.81	1,473,604.40	1,476,244.03	12,203.63	4.40
	116.00	5.12	1,454,101.33	1,438,704.28	12,203.63	4.99
	113.27	5.52	1,420,293.91	1,404,896.85	12,203.63	5.38
	112.15	5.31	1,406.405.77	1,391,008.72	12,203.63	5.17
	118.91	2.06	1,489,956.10	1,493,183.08	12,203.63	2.02
	117.65	2.93	1,474,420.51	1,477,647.49	12,203.63	2.86
	120.77	3.10	1,513,016.66	1,516,243.64	12,203.63	3.03
1999	113.23	0.37	1,419,871.07	1,423,562.30	12,203.63	0.52
	112.58	1.92	1,411,746.61	1,415,437.85	12,203.63	2.05
	122.05	2.78	1,528,792.66	1,532,483.89	12,203.63	2.89
	117.42	2.84	1,471,635.44	1,467,721.85	12,203.63	2.33
	119.38	3.18	1,495,806.96	1,491,893.37	12,203.63	2.68
	123.34	4.90	1,544,800.71	1,540,887.12	12,203.63	4.38
	120.62	3.98	1,511,154.39	1,500,157.65	12,203.63	4.27
	118.75	4.84	1,488,002.94	1,477,006.21	12,203.63	5.13
	115.81	3.26	1,451,655.21	1,440,658.47	12,203.63	3.57
	124.23	4.48	1,555,787.23	1,559,485.12	12,203.63	4.44
	125.33	6.53	1,569,325.21	1,573,023.11	12,203.63	6.45
	126.68	4.89	1,586,083,52	1,589,781.42	12,203.63	4.85

Table 3. Forecasting results for Mexico's Real GDP (with 0 preliminary observations)

Year	Month	Forecasts	Standard error	Annual rate	Quarter	Forecasts	Standard error	Annual rate
2000	J	1,525,593.09	31,276.80	7.17				
	F	1,516,317.33	41,949.11	7.13				
	M	1,591,514.26	50,410.59	3.85	I	1,544,474.89	35,794.32	5.99
	A	1,526,872.83	57,696.51	4.03				
	M	1,548,306.93	64,112.22	3.78				
	J	1,569,888.96	69,941.88	1.88	II	1,548,356.24	60,230.91	3.21
	J	1,531,828.11	75,321.68	2.11				
	A	1,513,188.78	80,342.06	2.45				
	S	1,493,058.92	85,066.65	3.64	III	1,512,691.94	77,280.20	2.72
	O	1,600,319.31	89,542.31	2.62				
	N	1,594,810.54	91,401.85	1.39				
	D	1,614,403.47	93,224.17	1.55	IV	1,603,177.77	88,490.70	1.85
Total		–	–	–		1,552,175.21	58,491.73	3.41

Table 4. Forecasting results for Mexico's Real GDP (with 2 preliminary observations)

Year	Month	Forecasts	Standard error	Annual rate	Quarter	Forecasts	Standard error	Annual rate
2000	J	1,518,578.29	14,026.71	6.67				
	F	1,539,458.36	14,026.71	8.76				
	M	1,614,655.29	31,276.80	5.36	I	1,557,563.98	12,345.65	6.89
	A	1,550,013.86	42,022.71	5.61				
	M	1,571,447.97	50,471.75	5.33				
	J	1,593,029.99	57,696.51	3.38	II	1,571,497.28	45,439.88	4.75
	J	1,554,969.14	64,112.22	3.65				
	A	1,536,329.82	69,941.88	4.02				
	S	1,516,199.96	75,321.68	5.24	III	1,535,832.97	66,402.19	4.29
	O	1,623,460.35	80,342.06	4.10				
	N	1,609,994,60	85,066.65	2.35				
	D	1,629,587.53	89,542,31	2.50	IV	1,621,014.16	82,181.02	2.98
Total		–	–	–		1,571,477.10	45,940.68	4.69

Table 5. Results of the recursive disaggregation of Mexico's Real GDP (Quarter 2000: I)

Year/Month		IMGAE	Annual rate	Preliminary series	Disaggregated series	Standard error	Annual rate
2000	Jan	121.01	6.89	1,516,028.82	1,530,301.62	12,203.63	7.50
	Feb	122.70	8.99	1,536,908.89	1,551,181.69	12,203.63	9.59
	Mar	128.30	5.12	1,606,074.13	1,620,346.93	12,203.63	5.73

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