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On a Gerardi Alternative for the Geary-Khamis Measurement of International Purchasing Powers and Real Product

Salem H. Khamis[1] *and D. S. Prasada Rao*[2]

Abstract: This paper briefly examines an aggregation method due to Gerardi which was used by EUROSTAT in its international comparison exercises (EUROSTAT (1977)). The Gerardi method is based on a simple geometric mean of the national prices expressed in different currency units. This is justified by the claim that the final comparisons are not affected by whether the national prices are converted or not converted before averaging. In this paper, we establish algebraically that such a claim is not valid as it involves the cancellation of zero coefficients in the numerator and denominator of a certain fraction.

Key words: National prices; purchasing powers of currencies; Geary-Khamis method; Gerardi method; solution to linear equation systems; international comparisons.

1. Introduction

The Geary-Khamis (GK) method of aggregation (Geary (1958); Khamis (1970, 1972)) is now used by the United Nations International Comparisons Project (ICP), the Statistical Office of the European Communities (EUROSTAT), the Organisation for Economic Co-operation and Development (OECD), and the regional commissions of the United Nations for inter-country comparisons of purchasing powers and real product (Kravis, Heston, and Summers (1982); EUROSTAT (1983); and OECD (1982)). The Food and Agriculture Organisation of the United Nations (FAO) also applied the GK method in calculating regional and world indexes of food and agricultural production starting in 1986 (FAO (1986)). However, the first EUROSTAT comparison of the Common Market countries for the year 1975 is based on a method due to D. Gerardi (Gerardi (1974); EUROSTAT (1977)) which is basically an unweighted version of the GK method using

[1] 23 Hillfield Road, Hemel Hempstead, Herts, U.K. HP2 4AA.
[2] University of New England, Armidale, Australia.

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the simple geometric mean of national prices to define average prices for any commodity. The GK method defines the average prices P_i of N commodities for M countries and the corresponding exchange rates R_j through the system of $M + N$ linear homogeneous equations

$$P_i^{\mathrm{GK}} = \frac{\sum_{j=1}^{M} R_j p_{ij} q_{ij}}{\sum_{j=1}^{M} q_{ij}}, \quad i = 1, 2, \ldots, N \tag{1}$$

$$R_j = \frac{\sum_{i=1}^{N} P_i^{\mathrm{GK}} q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}}, \quad j = 1, 2, \ldots, M \tag{2}$$

where p_{ij} and q_{ij} are the price and quantity of commodity i for country j.

In general the system of equations (1) and (2) has a unique positive solution for the P_i and R_j apart from an undetermined scalar multiplicative parameter. The Gerardi method used for the 1975 EUROSTAT comparison replaces the weighted arithmetic means in equation (1) by the simple geometric mean

$$P_i^{\mathrm{G}} = \left(\prod_{j=1}^{M} p_{ij} \right)^{1/M}, \quad i = 1, 2, \ldots, N \tag{3}$$

with the corresponding definition of R_j^{G} similar to the GK equations (2), i.e.,

$$R_j^{\mathrm{G}} = \frac{\sum_{i=1}^{N} P_i^{\mathrm{G}} q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}}, \quad j = 1, 2, \ldots, M. \tag{4}$$

Objections to the use of a simple geometric average of national prices expressed in different national currencies were dismissed by the claim that had these prices first been converted to a common currency, the comparisons of real product and purchasing powers would not have been affected. Had the prices been converted before averaging, then the Gerardi equations (3) and (4) would have to be replaced by what we call the Gerardi-type equation system,

$$P_i^{\mathrm{G}_1} = \left[\prod_{j=1}^{M} R_j^{\mathrm{G}_1} p_{ij} \right]^{1/M}, \quad i = 1, 2, \ldots, N \tag{5}$$

$$R_j^{\mathrm{G}_1} = \frac{\sum_{i=1}^{N} P_i^{\mathrm{G}_1} q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}}, \quad j = 1, 2, \ldots, M. \tag{6}$$

Equation (5) is the same as equation (5.6) in EUROSTAT (1982, p. 51) and equation (6) is the same as equations (2) and (4) above and also conforms with the definition of purchasing power p_h^A in equation (4) in EUROSTAT (1983, p. 40). The argument that equations (5) and (6) above lead to the same relative ratios as those of equations (3) and (4) is based solely on a well-known property of the geometric means of equations (3) and (5) alone which are claimed to allow the cancellation of the factor $\left[\prod_{j=1}^{M} R_j^{\mathrm{G}_1} \right]^{1/M}$ in the numerators and denominators of the ratios $P_i^{\mathrm{G}_1}/P_s^{\mathrm{G}_1}$ and $R_j^{\mathrm{G}_1}/R_k^{\mathrm{G}_1}$, thus leading to the same values as the ratios $P_i^{\mathrm{G}}/P_s^{\mathrm{G}}$ and $R_j^{\mathrm{G}}/R_k^{\mathrm{G}}$ obtained from the system of equations (3) and (4). This argument appears to have been accepted as a justification for not converting the national prices to a common currency before averaging them (without first showing that the cancelled factors are different from zero) as illustrated, for example, in EUROSTAT (1982, p. 51).

The main purpose of this note is to show the fallacy in the justification of the use of a simple geometric mean of national prices without conversion to a uniform currency. This fallacy is due to the fact that the cancellation of the related product of R_j's is not valid because this factor is equal to zero. In other words, it is shown below that the system of equations (5) and (6) has only the trivial solution $P_i^{\mathrm{G}_1} = R_j^{\mathrm{G}_1} = 0$ for all i and j.

2. Solution of the Gerardi-type Equation System

We consider the Gerardi-type equations (5) and (6) and for simplicity we drop in this section the superscript G_1 from the equations. We first observe that these equations have the trivial solution $P_i = R_j = 0$ for all i and j. We show now that, generally, this is the only solution to this set of equations. For this purpose we substitute for P_i in equation (6) its value from equation (5) to obtain, after simple algebra, for each j

$$R_j = \sum_{i=1}^{N} \frac{1}{p_{ij}} \left(\prod_{t=1}^{M} R_t p_{it} \right)^{1/M} w_{ij}, \qquad (7)$$

where

$$w_{ij} = \frac{p_{ij} q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}}. \qquad (8)$$

Assuming $R_j \neq 0$ for all j, we may divide both sides of equation (7) by R_j and, after simple manipulation, obtain

$$\left(\prod_{t=1}^{M} \frac{R_t}{R_j} \right)^{1/M} A_j = 1, \quad j = 1, 2, \ldots, M \qquad (9)$$

where

$$A_j = \sum_{i=1}^{N} \left(\prod_{t=1}^{M} \frac{p_{it}}{p_{ij}} \right)^{1/M} w_{ij}.$$

The value of A_j, being independent of the exchange rates R_t, can be directly calculated for all j from the national price and quantity data. Taking logarithms of both sides of equations (9) we obtain

$$\frac{1}{M} \sum_{t=1}^{M} (\log R_t - \log R_j) + \log A_j = 0,$$
$$j = 1, 2, \ldots, M \qquad (10)$$

which in matrix form is equivalent to

$$Ax = b \qquad (11)$$

where

$$A = \begin{bmatrix} -\frac{(M-1)}{M} & \frac{1}{M} & \cdots & \frac{1}{M} \\ \frac{1}{M} & -\frac{(M-1)}{M} & \cdots & \frac{1}{M} \\ \vdots & \vdots & & \vdots \\ \frac{1}{M} & \frac{1}{M} & \cdots - & \frac{(M-1)}{M} \end{bmatrix} \qquad (12)$$

is a square matrix of order M. Furthermore,

$$X = (\log R_1, \log R_2, \ldots, \log R_M)^T \qquad (13)$$

and

$$b = (-\log A_1, -\log A_2, \ldots, -\log A_M)^T \qquad (14)$$

are two column vectors of order M each, and T denotes transpose. Since the sum of each of the columns in A is zero, the rank of A is at most $M - 1$. The determinant of the submatrix obtained by deleting the first row and first column has a strictly dominant diagonal and hence the rank of the matrix A is exactly $M - 1$.

Accordingly the matrix equation (11) will have a solution if the rank of the augmented matrix $(A\ b)$ is equal to that of A, where $(A\ b)$

is

$$\begin{bmatrix} -\frac{(M-1)}{M} & \frac{1}{M} & \cdots & \frac{1}{M} & -\log A_1 \\ \frac{1}{M} & -\frac{(M-1)}{M} & \cdots & \frac{1}{M} & -\log A_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{M} & \frac{1}{M} & & -\frac{(M-1)}{M} & -\log A_M \end{bmatrix}.$$

This is an $M \times (M+1)$ matrix, whose rank is equal to that of A (i.e., $M-1$) if and only if $\sum_{j=1}^{M} \log A_j$ is equal to zero, which is generally not satisfied. This result can be proved by observing that the rank of $(A\ b)$ will be equal to $(M-1)$ if and only if there exists a non-trivial linear combination of the rows of $(A\ b)$ resulting in a zero row vector, and that the only linear combination with such a property is simply the sum, or a multiple of the sum, of the rows of the matrix.

Accordingly, the equations (5) and (6) are generally inconsistent and have only the trivial solution $R_j = P_i = 0$ for all i and j. A numerical example illustrating this result is provided in the Appendix.

3. Conclusion

The non-existence of a solution to the Gerardi-type equations (5) and (6) other than the trivial solution invalidates the argument of its being equivalent to the system of equations (3) and (4) as this argument involves the cancellation of zero factors from the numerators and denominators of ratios of the form R_j/R_k or P_i/P_s. Some objections to the use of Gerardi's equation system (3) and (4) are given in Kravis, Heston, and Summers (1982, pp. 78–79) and EUROSTAT (1982, pp. 59–61). We are not concerned here with the advantages or disadvantages of any particular system for international comparisons. We should also point out that the Gerardi System of equations (3) and (4) does lead to comparisons of real product and purchasing powers comparable numerically with those obtained by other systems of index numbers. All we assert here is that justification of averaging national prices before expressing them in terms of a uniform currency is still lacking.

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Appendix

A Numerical Illustration

We consider a simple example with three countries ($M=3$) and two commodities ($N=2$). The numbers used in this example are fictitious and the example is intended purely as an illustration of the main conclusion of the paper. The price and quantity data are given below:

	Country j					
Commodity i	1		2		3	
	Price	Quantity	Price	Quantity	Price	Quantity
1	1	10	3	4	5	2
2	2	5	4	6	6	5

The associated value share matrix is:

$$W=(w_{ik})=\begin{bmatrix}0.5 & 0.33 & 0.25\\ 0.5 & 0.67 & 0.75\end{bmatrix}$$

Following equations (9), A_k is given by

$$A_k=\sum_{i=1}^{N}\left[\prod_{j=1}^{M}\left(\frac{p_{ij}}{p_{ik}}\right)^{1/M}\right]w_{ik}, \quad k=1,2 \text{ and } 3.$$

Numerical values of A_1, A_2 and A_3, for the price-quantity data above, are

$$A_1=2.123 \quad A_2=0.8812 \quad A_3=0.5808.$$

Therefore,

$$A_1A_2A_3=1.0866$$

and

$$\log A_1+\log A_2+\log A_3=0.083.$$

In this case the condition that $\sum_k \log A_k$ should be equal to zero, necessary for the existence of a nontrivial solution to the Gerardi-type equations (5) and (6), is violated. In general most data sets would result in non-zero values for $\sum_k \log A_k$ unless the expenditure ratios, w_{ik}, are identical across all the countries.