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Optimal Dynamic Sample Allocation Among Strata

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A dynamic sampling plan among strata is a permutation of sampled items specifying which stratum is to receive the next item to be included in the sample. An optimal such plan has the property of achieving minimum variance for its cost whenever it is truncated. This article shows that optimal dynamic sampling plans exist under very general conditions, and gives a simple algorithm for constructing them. The well-known Neyman allocation is compared to the dynamic optimum, both statically and dynamically. The theorems rely on a version of the Neyman-Pearson Lemma adapted for use in search theory.

Key words: Dynamic sampling plan; Neyman allocation; sampling plan; search theory; static sampling plan; strata.

1. Introduction

In planning for a stratified sample in the context of an audit of a governmental data base, it was not too hard to guess variances and relative costs of the strata. However, absolute costs depended on how difficult it would be to find the hardcopy substantiation for sampled items. When the plan was made, we had very little information about this. It was entirely plausible that our cost estimates could be wrong by a constant factor of 2 (up or down), and not implausible that they could be wrong by a factor of 5. Thus we could guess the ratio of stratum sampling costs, but translating those into money or auditor time involved much greater uncertainty. Using standard Neyman allocation in this context could result in a sampling plan that is seriously over budget if sampling items is more time-consuming than anticipated. On the other hand, it could result in a plan that has larger variance than necessary because it failed to utilize fully the available sampling budget, if sampling items is less time-consuming than anticipated. If there were only a single stratum, a random permutation of the items, together with instructions to examine items in the order specified by the permutation and to examine as many as resources allow, results in a valid random sample regardless of where in the permutation sampling stops, provided that stopping is independent of the sample results. The problem addressed in this article is to find an analogous sampling plan when there are several strata.

Such a sampling plan is called "dynamic," in contrast to "static" plans based on either a fixed maximum budget or a fixed minimum variance to be attained. A dynamic sampling plan should optimally have the property that wherever it is stopped, the result is an optimal

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allocation of sample units among strata. It is not obvious that such a sampling plan exists. I show here that such plans do exist, give a simple necessary and sufficient condition for a sampling plan to be dynamically optimal, and give a simple algorithm for constructing such a plan. The analysis in this article draws heavily on that in Kadane (1968), using an analogy between the jth search of box h and the assignment of j items to stratum h in a sampling plan.

The remainder of this article is organized as follows: Section 2 reviews the literature and states the main result of the article, Section 3 compares the resulting algorithm to Neyman allocation, Section 4 gives practical considerations and an example, and Section 5 concludes. The theorems and proofs are in the Appendix.

2. Previous Literature and the Current Result

The current method of allocating sample observations to strata is to take sample sizes in strata proportional to the number of items in the stratum times the stratum standard deviation divided by the squared root of the cost of sampling per stratum element. The result that this allocation minimizes cost for a fixed target variance, or minimizes variance for a fixed target cost, is known as Neyman allocation, after Neyman's (1934) famous paper. Zarkovich (1956, 1962) shows that Tschuprow (1923a and b) and Kowalsky (1924) anticipated Neyman in this respect. Fienberg and Tanur (1995, 1996, 2001) show that Gram (1883) in turn anticipated Tschuprow and Kowalsky.

Neyman allocation involves a double approximation. A fundamentally discrete problem, the choice of the number of items in a stratum to be sampled, is replaced by a continuous one, in which it is imagined to be possible to allocate noninteger numbers of observations to a stratum. The result cited above is then the solution to the continuous problem. Then this continuous solution is in turn approximated by integer allocations. However, as shown for example in Cochran (1977, pp. 115 ff.), the optimum is quite flat in that small deviations in the allocations cause small variations in the consequent variance obtained. Therefore, for the problem for which it is designed, Neyman allocation has worked well in practice.

The approach taken here eliminates both approximations, and delivers a permutation of the union of the strata having the property that wherever it is stopped, the resulting allocation has the smallest variance possible among allocations costing no more than it does. Furthermore the required computations are very simple.

To be precise, a static sampling plan is a sample from the universe of items. After they are selected, sampling ceases. An optional static sampling plan minimizes some loss function, such as a variance, subject to a constraint on the resources it requires. A dynamic sampling plan is a permutation (i.e., an ordered list) of the universe of items, together with the instruction to sample in the order specified by the permutation until sampling resources are exhausted, or until variances are sufficiently small. An optimal dynamic sampling plan is an optimal static sampling plan wherever it is stopped.

The mathematics in the Appendix can be summarized in the following result:

Theorem 1: Let $\Delta_{j,h}$ be the reduction in variance occasioned by taking j instead of $j-1$ items from stratum h in the sample, and let $c_{j,h}$ be the added cost of doing so. Suppose that $\Delta_{j,h}/c_{j,h}$ is strictly decreasing in j for each h.

Then a sampling plan that starts with the allocation of one sampled item to each stratum, and then orders allocations by $\Delta_{j,h}/c_{j,h}$, highest first (breaking ties arbitrarily), is dynamically optimal, and only such sampling plans are dynamically optimal.

In Theorem 1, $\Delta_{j,h}/c_{j,h}$ is a benefit-cost ratio. $\Delta_{j,h}$ is the decrease in variance in assigning j instead of $j-1$ sample points to stratum h. Similarly $c_{j,h}$ is the additional cost of doing so.

To apply this result to the standard case sampling to estimate the mean of a stratified sampling under linear cost, I use the following notation:

Suppose there are K strata indexed by h, and $j = 1, \ldots, \infty$ indexes the number of allocations to a stratum. A static sampling plan specifies the number of items n_h to be sampled from stratum h. The variance of the sampling plan with n_h of the sample allocated to stratum h is

$$V(\bar{y}_{st}) = \frac{1}{N^2}\sum_{h=1}^{K}\frac{N_h^2 S_h^2}{n_h} - \frac{1}{N^2}\sum_{h=1}^{K} N_h S_h^2 \tag{1}$$

where S_h^2 is the variance of stratum h, N_h is the number of elements in stratum h and $N = \sum_{h=1}^{K} N_h$. See Cochran (1977, p. 97). Note that $V(\bar{y}_{st})$ is infinite if any $n_h = 0$, so we restrict ourselves to sampling plans with $n_h \geq 1$. The cost of such a sampling plan is taken to be

$$C(\bar{y}_{st}) = C_0 + \sum_{h=1}^{K} n_h c_h \tag{2}$$

It should be noted that these costs are the anticipated costs in the design stage. Actual costs may differ from these by some constant multiple. Note, however, that the dynamically optimal sampling plan is invariant to the value of that constant multiple.

Let $d_h = N_h^2 S_h^2 / N^2$. Then the decrease in variance occasioned by increasing the allocation to stratum h from $j-1$ to j is

$$\Delta_{j,h} = \frac{d_h}{j-1} - \frac{d_h}{j} = \frac{d_h}{j(j-1)} \tag{3}$$

provided $j > 1$, and the cost increases by

$$c_{j,h} = c_h \tag{4}$$

Since $\frac{\Delta_{j,h}}{c_{j,h}} = \frac{d_h}{c_h(j)(j-1)}$ strictly decreases in j for each h, a dynamically optimal sampling plan exists, and is given by the theorem above.

This method can be applied to other loss functions than the variance of a sample mean, and to other cost functions. For example, Cochran (1977, p. 96) mentions the possibility of a cost function in which travel costs dominate, and are approximated by

$$\sum_{h=1}^{K} t_h \sqrt{n_h}$$

Then $c_{j,h} = t_h[\sqrt{j} - \sqrt{j-1}]$. The key condition that assures the existence of a dynamically optimal sampling plan is that $\Delta_{j,h}/c_{j,h}$ is decreasing in j for each h. In this case,

$$\frac{\Delta_{j,h}}{c_{j,h}} = \frac{d_h}{t_h j(j-1)(\sqrt{j} - \sqrt{j-1})}$$

To see that this is decreasing in j, observe that $(\sqrt{j} - \sqrt{j-1})(\sqrt{j} + \sqrt{j-1}) = 1$. Hence

$$\frac{\Delta_{j,h}}{c_{j,h}} = \frac{d_h}{t_h} \cdot \frac{\sqrt{j} + \sqrt{j-1}}{j(j-1)} = \frac{d_h}{t_h}\left[\frac{1}{j^{1/2}(j-1)} + \frac{1}{j(j-1)^{1/2}}\right]$$

which decreases as j increases.

Hence, using this $\Delta_{j,h}$ and $c_{j,h}$, the same results and algorithm may be applied to find a dynamically optimal sampling plan, and hence optimal sampling plans for fixed variances or fixed cost.

3. A Comparison with Neyman Allocation in the Static Case

As mentioned above, Neyman Allocation amounts to minimizing (1) by choice of the numbers n_h, subject to (2) and the constraints

$$n_h \le N_h \ \text{ for each stratum } \ h = 1, \ldots, K \tag{5}$$

Typically solutions to this minimization neglect the constraints (5) until later.

This minimization, for real numbers n_h, can be regarded as a nonlinear program. Cochran (1977, pp 97, 98) solves it using the Cauchy-Schwartz inequality, following Stuart (1954). It can also be thought of as a geometric program (Duffin et al. 1967). I prefer to derive it using a LaGrange multiplier technique, as this is more general, applying to loss functions other than (1) and cost constraints other than (2).

In the notation introduced above, (1) can be rewritten as

$$V(\bar{y}_{st}) = \sum_{h=1}^{K} \frac{d_h}{n_h} - L$$

where $L = \frac{1}{N^2}\sum_{h=1}^{K} N_h S_h^2$ does not depend on the allocations n_h. Then forming the Lagrangian,

$$\mathscr{L} = \sum_{h=1}^{K} \frac{d_h}{n_h} - L + \lambda\left(\sum_{h=1}^{K} c_n N_h + C_0 - C\right)$$

Taking the first derivative with respect to n_h and λ and setting them equal to 0, yields, respectively,

$$\frac{-d_h}{n_h^2} + \lambda c_h = 0 \quad h = 1, \ldots, K \tag{6}$$

and (2).

Rearranging (6) gives

$$n_h = \frac{1}{\sqrt{\lambda}}\sqrt{\frac{d_h}{c_h}} \quad h = 1, \ldots, K \tag{7}$$

Now the constraint (2) can be used to solve for $1/\sqrt{\lambda}$:

$$C - C_0 = \sum_{h=1}^{K} c_h n_h = \frac{1}{\sqrt{\lambda}}\sum_{h=1}^{K} c_n\sqrt{d_n/c_h} = \frac{1}{\sqrt{\lambda}}\sum_{h=1}^{K}\sqrt{c_n d_h}$$

Hence $\frac{1}{\sqrt{\lambda}} = \frac{C-C_0}{\sum_{h=1}^{K}\sqrt{c_n d_h}}$. Substituting this result into (7) yields

$$n_h = \frac{(C - C_0)\sqrt{d_n c_h}}{\sum_{h=1}^{K}\sqrt{c_h d_h}} \tag{8}$$

So far, the constraints (5) have been neglected. When (8) yields a solution violating (5) for some strata, those strata are to be sampled fully, by setting $n_h = N_h$, and (8) is to be recomputed for the remaining strata, adjusting C_0 and K accordingly. This results in allocations to the remaining strata higher than previously found. Again (5) may be violated. This process is to be repeated until (5) is satisfied for the remaining strata.

Finally, the constraint that the n_h's be integers must be addressed. There may not exist an integer solution to the minimization of (1) subject to (2), so (2) must be replaced by

$$C(\bar{y}_{st}) \geq C_0 + \sum_{h=1}^{K} n_h c_n \tag{9}$$

The fully sampled strata already have allocations n_h that are integers, so they need not be considered further. The less-than-fully-sampled strata have not-necessarily-integer allocations from (8) that can be thought of as consisting of an integer part and a fractional part, i.e.,

$$n_h = i_h + f_h \tag{10}$$

where i_h is an integer, and $0 \leq f_h < 1$ is the fractional part. Now a new minimization results, namely to minimize, with respect to f_h,

$$\sum_{h=1}^{K}\frac{d_h}{i_h + f_h} - L \tag{11}$$

subject to the constraints

$$\sum c_h f_h \leq 0 \tag{12}$$

$$\text{and } f_h \text{ integers.} \tag{13}$$

This nonlinear integer programming problem is usually avoided by applying some heuristic rounding rule to the allocations derived in (8). As mentioned earlier, because of the flatness of the variance function in the variables n_h, these heuristic rules tend to work satisfactorily in practice.

The minimization technique used in Theorem 1 avoids the complication of re-computation if there are fully-sampled strata and automatically gives integer solutions. It gives all the possible combinations of variances achieved and cost expended, allowing a statistician to choose that combination most suitable to the problem at hand.

None the less, it is reasonable to suppose that the solutions proposed by Theorem 1 and those given by Neyman Allocation will approximate each other under certain conditions. Note that the ratios $\Delta_{j,h}/c_{j,h} = \frac{d_h}{c_h(j)(j-1)}$ approach 0 as j gets large. Suppose that the allocation algorithm specified by Theorem 1 has been going for some time, so that each stratum has some minimal allocation. Limiting attention to those strata not yet sampled with certainty, to a reasonable approximation the ratios $\Delta_{j,h}/c_{j,h}$ will be the same for each such stratum h. This will not be exactly true, of course, but because strata with current ratios substantially larger than others would have been included in the current allocations already, it will be approximately true. Thus the n_h's for strata not yet sampled with certainty will satisfy roughly

$$\frac{\Delta_{n_h,h}}{C_{n_h,h}} \sim s \tag{14}$$

for some constant s.

The second approximation I make is that, again roughly, $\frac{1}{n_n(n_h-1)} \sim \frac{1}{n_h^2}$. For example, when n_h is 10, $\frac{1}{90} = .0111$ is not very different from $\frac{1}{100} = .01$, and the difference will decline even further as n_h grows.

Combining these approximations yields

$$s \sim \frac{\Delta_{n_h,h}}{c_{n_h,h}} = \frac{d_h}{c_h(n_h)(n_h-1)} \sim \frac{d_h}{c_h n_h^2} \tag{15}$$

Solving for n_h,

$$n_h \sim \frac{1}{\sqrt{s}}\sqrt{\frac{d_h}{c_h}} \tag{16}$$

which is the Neyman allocation. Thus it is reasonable to expect that the allocation of Theorem 1 will differ from those of Neyman allocation most strongly when stratum sizes are fairly small, as this is when the integer restriction matters the most.

4. Practical Considerations

In application, it is important that each stratum be given a random permutation, so that each element of each stratum has the same chance of being in each position in the dynamically optimal allocation to which an element of that stratum is assigned.

When sampling is intense, so that large numbers of observations are assigned to each stratum, it is to be expected that the continuous approximations involved in the Neyman allocation will be less onerous. Conversely, the optimal allocation will be relatively farther from the Neyman allocation when the samples are small. This point is illustrated in Figure 1, computed in an example with three strata, with respective sizes 3, 5 and 7, variances 7, 9, and 11, and costs per sample 2.5, 4.2, and 6.7. The most desirable direction is toward the lower left, combining small cost with small variance. The line gives the combinations of costs and variances from the Neyman allocation, but these cannot be attained. The dots are from the dynamic optimum, and can be attained. The reason for the difference is that the Neyman allocation assumes that any real number can be used as a stratum allocation, so it achieves apparently better, but unobtainable, combinations of cost and variance. By contrast, the dynamic allocation is restricted only to integer allocations. Notice that toward the upper left of the graph, indicating a larger sample than the lower right, the dots and the curve move closer together, as hypothesized.

A program in S + to compute dynamic allocations has been submitted to StatLib's archive (http://lib.stat.cmu.edu/S).

An important qualification to any dynamic sampling plan is that the stopping rule is independent of the sampling results. Thus if the people doing the sample look ahead to see which items are next, and use prior information about likely results if the next items were sampled, the validity of the result is destroyed. Alternatively, it might be the case that sample results are not independent of sample costs, if, say, a discrepant result in an audit costs more to research than a nondiscrepant one. These cases are outside the scope of this article.

I envisage that standard analysis can be used, conditional on the stratum allocations n_h. In the lead example of estimating a population mean, the stratified sampling variance (1) would still be relevant, conditional on the n_h's actually obtained.

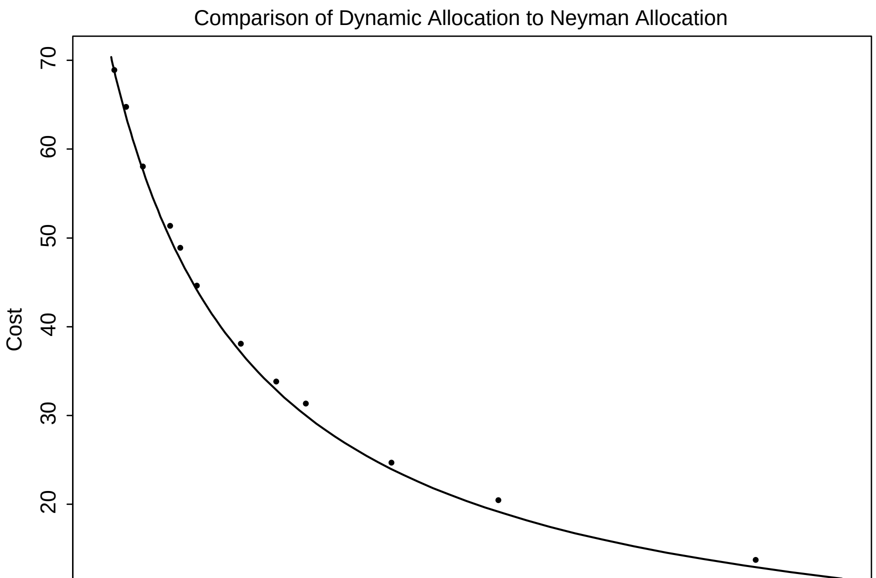

Figure 1. The continuous line gives the Neyman allocation costs and variances; the dots are the dynamic allocation costs and variances. For example, the dot on the lower right corresponds to $n_1 = n_2 = n_3 = 1$ with cost $(1)(2.5) + (1)(4.2) + (1)(6.7) = 13.4$

While uncertainty about a constant factor multiplying costs was the motivating example for this article, uncertainty about variances is also a concern. While they are assumed known in the design phase, of course they are estimated after the data are gathered. A referee writes "In my experience, as the sample size increases, apparent variance may also increase substantially, due to an increase in nonsampling error (i.e., a decrease in data quality). In the application described, this could just mean that care should be taken in the collection of each additional observation, rather than rushing to see how many observations can be made." Additionally, it is worth noticing that the dynamically optimal plan is invariant to misestimation of variances by a common constant, just as it is to misestimation of costs by a common constant.

5. Conclusion

This article draws a distinction between static and dynamic sampling plans. In a static plan, either the maximum cost allowed or the minimum variance required is known in advance. In such a situation, Neyman allocation generally works satisfactorily. However, in a dynamic problem, in which the plan has to be flexible in case the costs are either too high or too low by a constant factor, Neyman allocation is awkward and ill-suited. Optimal dynamic plans are shown to exist, and are easy to compute.

Appendix

Thus we seek a strategy to maximize

$$\sum \Delta_{j,h} \tag{17}$$

subject to

$$\sum c_{j,h} \leq C - C_0 \equiv L \tag{18}$$

where both summations extend over all (j, h) for which there is a jth element from stratum h in the sampling plan.

Because (5) and (6) do not depend on the order in which the sample units in σ are examined, the double summation over j and h can be replaced with a single summation over i, where i represents a pair (j, h), and i is an element of an unordered set S. However, an additional constraint must be imposed, that if a set includes assignment (j, h), it must include assignment $(j - 1, h)$ for all $j > 1$. Such a set is called feasible.

Hence we seek a set S maximizing

$$\sum \Delta_i \tag{19}$$

subject to the constraint

$$\sum c_i \leq L \tag{20}$$

where both summations extend over S. The assumption of feasibility is ignored for the moment; later it will be shown that the sets maximizing (7) subject to (8) are feasible.

This maximization of (7) subject to (8) is very close to that solved by the Neyman-Pearson lemma, in which a critical set is chosen to maximize power subject to the constraint that the size of the test be no larger than some specified α. Let B be the sum of costs c_i over those Δ_i that are positive, and suppose $C_0 < L < B$, so that some, but not all useful sampling is included in the budget. Note that B might be infinite.

The following theorem is a version of the Neyman-Pearson Lemma (Kadane 1968, Theorem 1; Lehmann 1959, pp. 65, 66):

Theorem 2: Let $\{\Delta_i\}$ and $\{c_i\}$ be arbitrary nonnegative sequences such that $\sum \Delta_i < \infty$. Let X be the class of sequences x_i such that $0 \leq x_i \leq 1$ for all i. If $C_0 < L < B$, then the maximum of

$$\sum x_i \Delta_i \tag{21}$$

subject to

$$\sum x_i c_i \leq L \tag{22}$$

and $x \in X$ is attained, and it occurs when and only when

$$x_i = \begin{cases} 1 & \text{if } \Delta_i > rc_i \\ 0 & \text{if } \Delta_i < rc_i \end{cases} \tag{23}$$

for some r, $0 < r < \infty$, and

$$C_0 + \sum x_i c_i = L \tag{24}$$

The set of r's satisfying (11) is the same for each optimal x and is a single point or a closed interval.

The function X can be taken to have only the values 0 and 1 if and only if L can be expressed as the sum of cost of all stratum allocations with Δ_i/c_i bigger than some r, possibly together with costs of some stratum allocations with $\Delta_i/c_i = r$. Otherwise a fractional X must be used for some i, but one such x_i is always enough.

When x_i is neither 0 or 1, it can be interpreted as a randomization probability, in which with probability x_i the extra observation is included, and with probability $1 - x_i$ it is not. This has the same role as randomization to achieve a fixed level of a test in the Neyman-Pearson theory. With such a possible randomization, (9) can be thought of as the expected variance, and the left side of (10) as the expected cost.

Call a set of stratum allocations locally optimal if inclusion of i' and exclusion of i implies

$$\frac{\Delta_{i'}}{c_{i'}} \geq \frac{\Delta_i}{c_i} \tag{25}$$

By Theorem 2, a set of stratum allocations included or partially included in a sampling plan can correspond to a solution to the relaxed problem only if it is locally optimal.

Returning to the double-subscript notation and the requirement of feasibility, a set of sample allocations is locally optimal if inclusion of (j', h') and exclusion of (j, h) implies

$$\frac{\Delta_{j',h'}}{c_{j',h'}} \geq \frac{\Delta_{j,h}}{c_{j,h}} \tag{26}$$

Since $\Delta_{j,h}/c_{j,h}$ is strictly decreasing in j for each h, every locally optimal sampling allocation is feasible.

Theorem 3: Permit a randomized last allocation and suppose that $\Delta_{j,h}/c_{j,h}$ is strictly decreasing in j for each h. Then any set that maximizes (5) subject to (6) and costs no more than L, $C_0 < L < B$, includes all stratum allocations for which

$$\frac{\Delta_{j,h}}{c_{j,h}} > r$$

for some r, excludes all those for which

$$\frac{\Delta_{j,h}}{c_{j,h}} < r$$

and includes enough of those with

$$\frac{\Delta_{j,h}}{c_{j,h}} = r$$

to spend exactly L. Each such set is feasible and maximizes (5) subject to (6). A randomized last allocation is unnecessary if and only if L is the cost of some locally optimal set.

Theorem 3 shows that ordering allocations according to $\Delta_{j,h}/c_{j,h}$, largest first, and breaking ties arbitrarily, yields a sampling plan that is dynamically optimal.

When a randomized last allocation is not permitted and L is not the cost of some locally optimal set, the optimal allocation may be found using a branch and bound algorithm (see Kadane (1968, Section 3) and Kolesar (1967)). In this case the optimal allocation need not be a truncation of a dynamically optimal plan.

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