Journal of Official Statistics
Vol. 7. No. 3, 1991, pp. 311–323

The Heligman–Pollard Formula as a Tool for Expanding an Abridged Life Table

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Abstract: Proposed in this paper is a new method of computing a complete life table from an abridged one. An estimation procedure is demonstrated for calculating the one-year probabilities of dying from the five-year ones given in an abridged life table. This question is primarily of great interest when the abridged life table is the only one that can be constructed on the basis of available empirical mortality data in a given country. The main tool of our expansion technique is the Heligman–Pollard formula. An evaluation of this technique and comparisons with two other expansion techniques are also provided here. Used for these purposes are Swedish national mortality data for the period 1976–1980 and for both sexes.

Key words: Age pattern of mortality; probability of dying; interpolation formulae.

1. Introduction

In demography, a common method of completely describing a country's mortality pattern is to present it as a life table. In a complete life table the data are presented for every single year of age. An abridged life table contains data tabulated in wider age intervals, most often five-year ones, except for the first five years, which are usually presented in two intervals, [0, 1) and [1, 5).

There may be two different reasons for preparing an abridged life table rather than a complete one. The first reason is that, for many purposes, an abridged life table is sufficiently accurate and more convenient to use. The second is that in countries with incomplete and unstable documentation of vital statistics, the quality of the data may not permit computation of a complete life table.

Thus, it often happens that only an abridged life table is available when a complete one is needed; hence it is an important problem to construct a full table, given the abridged one. A conventional method is to obtain the values of the survival function that are missing in the abridged life table through straightforward polynomial interpolation between those values that are given there. One such method and some other ones, using more complicated computations, are reviewed in Section 2. The purpose of the present paper is to present and evaluate the new technique. The main tool is an eight-parameter formula proposed by Heligman and Pollard (1980), which is described in Section 3.

In Subsection 4.1, we describe the main procedure for estimating the one-year probabilities of dying. It was first presented

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Acknowledgment: The author wants to express deep gratitude and appreciation to Jan Lanke for his essential advice during the course of the research and writing of this paper. Thanks are also in order for Daniel Thorburn for a suggestion which resulted in Subsection 4.2.

by Kostaki (1987) and also developed independently, within the MORTPAK and MORTPAK-LITE software packages published by the United Nations (1988a, 1988b). In Subsection 4.2, we present a new adjusting procedure which may be applied to the results of the main expansion technique. Section 5 illustrates the results of our applications using Swedish national mortality data and provides an evaluation of our technique. Finally, Section 6 provides some concluding remarks.

2. Conventional Methods for Expanding an Abridged Life Table

A complete life table can be constructed with the knowledge of any one of

$$d_x,\ l_x,\ q_x$$

where d_x is the number of deaths in the age interval $[x, x + 1)$, l_x is the number of survivors at exact age x out of the radix l_0, and q_x is the conditional probability of dying in the age interval $[x, x + 1)$ for a person alive at age x.

A widely used method of computing a complete life table having an abridged table as a starting point is to use interpolation techniques on the existing survival probabilities $l(x)$. An efficient and extensively used technique is a six-point Lagrangian interpolation formula (see Elandt–Johnson and Johnson 1980). This formula expresses each nontabulated value of $l(x)$ as a linear combination of six particular polynomials in x, each of degree five

$$l(x) = \sum_{i=1}^{6} \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} l(x_i). \qquad (2.1)$$

Here $x_1, x_2, \ldots, x_6$ are the tabular ages nearest to x. This technique provides good approximations for adult mortality; however, it is less accurate for the early childhood ages.

An alternative procedure for breaking abridged life tables into one-year age groups was developed by Reed. A description of this procedure is provided by Valaoras (1984) who used it in order to produce the official complete life tables for the Greek male and female populations. This technique is based on the five-year mortality rates

$$_5m_x = \frac{_5d_x}{_5M_x}$$

where $_5d_x$ represents the number of deaths in the age interval $[x, x + 5)$ and $_5M_x$ represents the mean population of the same ages. Then, the one-year probabilities of dying q_7, $q_{12}, q_{17}, \ldots$ are calculated by the use of the approximate formula

$$q_{x+2} = \frac{2\,_5m_x}{2 + {_5m_x}}. \qquad (2.2)$$

In order to estimate the complete set of q_x-values ($x \geqslant 5$), the formula

$$\frac{q_x}{K^x} = a + bx + cx^2 + dx^3$$

is used, where a, b, c, and d are parameters to be estimated by least squares using the calculated one-year probabilities of dying $q_7, q_{12}, q_{17}, \ldots$. The formula is fitted twice, first using q_7, q_{12}, q_{17}, and q_{22}, with $K = 0.989943$ and the fitted model is used to calculate q_x for $5 \leqslant x \leqslant 20$. For $x \geqslant 25$ a new set of estimates of a, b, c, and d is calculated using q_{22}, q_{27}, q_{32}, now with $K = 1.0251234$. This model is used to calculate the values of q_x for $x \geqslant 25$. Finally, for the ages 21 to 24, a linear combination of the two fitted equations is used in order to estimate the one-year q_x-values:

$$q_{21} = 0.8q'_{21} + 0.2q''_{21}$$
$$q_{22} = 0.6q'_{22} + 0.4q''_{22}$$
$$q_{23} = 0.4q'_{23} + 0.6q''_{23}$$
$$q_{24} = 0.2q'_{24} + 0.8q''_{24}$$

where the first and second terms in the right hand side of the above equations are the fitted values of q_x for K equal to 0.989943 and 1.0251234, respectively.

This technique is not adequate for approximating early childhood mortality (for $x < 5$) and its accuracy is poor at the earlier adult ages as well. Nevertheless, it provides an effective means of approximating later adult and senescent mortality.

Recently, Mode and Busby (1982) have developed a formula which represents the survival probabilities, $l(x)$, as a parametric function of age. They determine the values of $l(x)$ according to the expression

$$l(x) = \begin{cases} l_0(x), \ 0 \leqslant x \leqslant \delta_0 \\ l_0(\delta_0)l_1(x - \delta_0), \ \delta_0 \leqslant x \leqslant \delta_1 \\ l_0(\delta_0)l_1(\delta_1 - \delta_0)l_2(x - \delta_1), \ x \geqslant \delta_1 \end{cases}$$

where

$$l_0(x) = \exp(\alpha_0(\exp(-\beta_0 x) - 1))$$

$$l_1(x) = \exp\left(\frac{\beta_1\gamma_1^3}{3} - \alpha_1 x + \frac{\beta_1}{3}(x - \gamma_1)^3\right)$$

$$l_2(x) = \exp(-\alpha_2 x - \beta_2(\exp(\gamma_2 x) - 1)).$$

Here, $l_0(x)$ is the probability that an individual is alive at age x, $0 \leqslant x \leqslant \delta_0$ and δ_0 is a positive parameter to be chosen. Further, $l_1(x - \delta_0)$ is the conditional probability that an individual aged $\delta_0 \geqslant 0$, is alive at age x, where $\delta_0 \leqslant x \leqslant \delta_1$ and δ_1 is a positive parameter to be chosen. Finally, $l_2(x - \delta_1)$ is the conditional probability that an individual who survived to age δ_1 is alive at age $x \geqslant \delta_1$.

In addition to the two parameters δ_0 and δ_1, the formula contains eight parameters to be estimated by a nonlinear least-squares procedure. This formula can be useful for performing interpolations of abridged life tables. However, a technical difficulty limits its use. Attempts made by Mode and Busby (1982) to estimate the eight parameters simultaneously have been unsuccessful. They propose an alternative estimation procedure. They start by choosing permissible values of the two parameters δ_0 and δ_1 ($\delta_0 = 10$, $\delta_1 = 30$), and then they estimate the remaining parameters by the use of a nonlinear stepwise least-squares procedure, where at each step at most three parameters are estimated. However, the fact that they empirically determine the values of the two parameters δ_0 and δ_1 limits the flexibility of the model.

3. The Heligman–Pollard Model

A recent attempt to represent mortality over the course of the entire life span, using a single analytical expression, has been made by Heligman and Pollard (1980). The idea underlying the Heligman–Pollard (H&P) model is that the causes of death can be divided into three classes, namely those affecting childhood, early and middle adult life, and old age.

The mathematical function H&P suggest is given by the formula

$$\frac{q_x}{p_x} = A^{(x+B)^C} + D\exp(-E(\ln(x/F))^2) + GH^x \qquad (3.1)$$

where the right hand side is interpreted as $A^{B^C} + G$ for $x = 0$. Here q_x is the model probability that an individual who has reached age x will die before reaching age $x + 1$, while $p_x = 1 - q_x$. The quotient q_x/p_x thus represents the odds that an individual of age x will die before he attains age $x + 1$. The positive parameters A, B, C, D, E, F, G, and H are to be estimated. An interpretation of these parameters are given in Heligman and Pollard (1980).

The first term in (3.1), a rapidly decreasing exponential, reflects the fall in mortality at the infant and early childhood ages. This term dominates at these ages, while the other two make insignificant contributions to the total expression. The second term in (3.1), a function similar to the lognormal density, reflects the middle life mortality. It reflects the accident mortality for the male population as well as the accident and, in developing countries, puerperal mortality for the females, often referred to in the demographic literature as the accident hump. Finally, the third term in (3.1), an exponential term as that of Gompertz, reflects the exponential rise in mortality at the later adult ages, i.e., at the ages greater than 40.

Several applications of this formula on a wide variety of mortality experiences in Australia (Heligman and Pollard 1980), in the U.S.A. (Mode and Busby 1982), and in Sweden (Hartmann 1987) have shown that the model provides quite a satisfactory representation of the age pattern of mortality. In a previous paper (Kostaki 1985), we applied this formula to Swedish and Greek national mortality schedules. Also in our applications the model has proved very efficient in describing the age pattern of mortality, providing very close fits to these mortality experiences.

In order to fit the H&P formula to the empirical q_x-values in a complete life table, the parameters of the model can be estimated by least squares. In all the applications mentioned above the sum of squares to be minimized has been taken as

$$\sum_x \left(\frac{\hat{q}_x}{q_x} - 1 \right)^2 \tag{3.2}$$

where $\hat{q}_x$ is the model probability that a person who has reached age x will die before reaching the age $x + 1$, and q_x is the corresponding empirical quantity. This version of the classical least-squares procedure has been proposed by Heligman and Pollard (1980). The expression (3.2) can also be regarded as the weighted sum of squares

$$\sum w_x(\hat{q}_x - q_x)^2$$

where $w_x = 1/q_x^2$.

4. Application of Heligman–Pollard Model to Expansion of Abridged Life Tables

4.1. The main procedure

Let us use the notation $\mathbf{C}$ for the coefficients in the H&P formula,

$$\mathbf{C} = (A, B, \ldots, H)$$

and let us use $F(x; \mathbf{C})$ to denote the right hand side of the H&P expression (3.1) for q_x/p_x. Then our technique for expanding an abridged life table can be described as follows.

From the model

$$\frac{q_x}{p_x} = F(x; \mathbf{C}) \tag{4.1}$$

for the one-year odds of dying, we get

$$q_x = \frac{F(x; \mathbf{C})}{1 + F(x; \mathbf{C})} = G(x; \mathbf{C}) \tag{4.2}$$

say, and hence the relation

$${}_nq_x = 1 - \prod_{i=0}^{n-1} (1 - q_{x+i}) \tag{4.3}$$

implies the following model for the death probabilities in the abridged life table

$${}_nq_x = 1 - \prod_{i=0}^{n-1} (1 - G(x + i, \mathbf{C})) = {}_nG(x; \mathbf{C})$$

say, where ${}_nG(x; \mathbf{C})$ is an explicit but complicated function of $\mathbf{C}$, x, and n.

Given an abridged life table one starts by estimating $\mathbf{C}$ through minimization of

$$\sum_x \left(\frac{{}_nG(x; \mathbf{C})}{{}_nq_x} - 1 \right)^2 \tag{4.4}$$

where the summation is over all relevant values of x (and where, in fact, n may depend on x). Then one inserts this $\mathbf{C}$ into the function $G(x; \mathbf{C})$, and takes as expanded life table the one containing the probabilities of dying so calculated.

4.2. An additional adjustment

The one-year probabilities of dying, $\hat{q}_x$, constructed by our expansion procedure can be described as that set of Heligman–Pollard probabilities whose corresponding n-year probabilities

$$1 - \prod_{i=0}^{n-1} (1 - \hat{q}_{x+i}) \tag{4.5}$$

most closely approximate those, say ${}_nq_x$, in the abridged life table that was our point of departure. However, it will not be true that the quantities (4.5) agree exactly with ${}_nq_x$. If that is considered as a drawback, a simple adjustment can be made.

If, for a certain x, the quantity defined by (4.5) is, say, smaller than the original ${}_nq_x$, then $\hat{q}_x, \hat{q}_{x+1}, \ldots, \hat{q}_{x+n-1}$ are, at least on the average, too small and it seems natural to increase them in some way to $\hat{q}'_x, \hat{q}'_{x+1}, \ldots, \hat{q}'_{x+n-1}$ so as to make

$$1 - \prod_{i=0}^{n-1} (1 - \hat{q}'_{x+i}) = {}_nq_x. \tag{4.6}$$

There are clearly several possible ways of doing that; the simplest is to choose

$$\hat{q}'_{x+i} = 1 - (1 - \hat{q}_x)^K$$

where

$$K = \frac{\ln(1 - {}_nq_x)}{\sum_{i=0}^{n-1} \ln(1 - \hat{q}_{x+i})}.$$

It is a simple matter to check that the $\hat{q}'_{x+i}$ so defined satisfy (4.6). It is also easy to explain the rationale behind this particular choice of adjustment. It amounts to assuming that the force of mortality $\mu'(x)$ underlying the original abridged life table is, in each n-year age interval $[x, x + n)$, a constant multiple, say $K\mu(\cdot)$, of one, say $\mu(\cdot)$, of those infinitely many forces of mortality which produce the complete life table that we obtained through our expansion process.

5. Some Comparisons

In order to evaluate our technique, we chose as a test material the empirical one-year probabilities of dying q_x, for the male and female populations of Sweden, for the period 1976–1980. The data are taken from Statistics Sweden (1984).

We started by computing the abridged ${}_nq_x$-values using the formula (4.3). Then we fitted the H&P model to these ${}_nq_x$-values. The sum of squares to be minimized was (4.4). A nonlinear least squares algorithm with the capability of approximating numerically all derivatives was used in order to estimate the parameters of the model. This algorithm (E04FDF), part of NAG library of computer programs, is based upon a modification of the Gauss–Newton iteration procedure. A detailed description of the algorithm is provided in Gill and Murray (1978). Displayed in Table 5.1 are the resulting parameter estimates.

By inserting these estimates into (4.2) we calculated the one-year probabilities of dying. Tables A1 and A2 in Appendix illus-

Table 5.1. Parameter estimates

	Males	Females
A	0.0005893	0.0004600
B	0.0043836	0.0047785
C	0.0828424	0.0801688
D	0.000706	0.000185
E	9.927863	12.968394
F	22.197312	19.515942
G	0.00004948	0.00003236
H	1.10003	1.09534

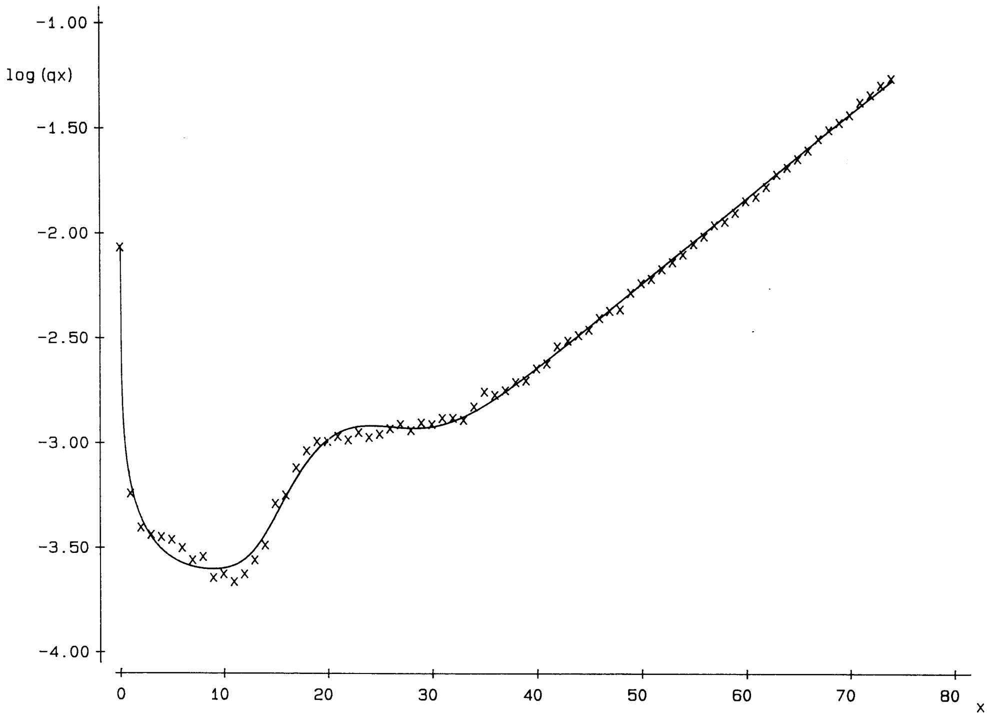

Fig. 1. Empirical q_x-values for Swedish males, 1976–80 (crosses) and computed q_x-values, main expansion technique (solid line)

trate the empirical q_x-values (column 1) and the corresponding calculated $\hat{q}_x$-values (column 2), for males and females respectively. The results of our computations are also presented graphically in Figures 1 and 2. These results indicate a very close agreement between q_x and $\hat{q}_x$ in both cases. In the vast majority of ages, the empirical and the calculated q_x-values agree to four decimal places.

Then, we applied our adjusting procedure to the $\hat{q}_x$-values as described in Subsection 4.2. The resulting $\hat{q}'_x$-values are presented in column 3 in Tables A1 and A2 for males and females respectively. In Figures 3 and 4, we illustrate the results of our computations. Comparing these figures with Figures 1 and 2, we can easily observe that the new adjusted sets of q_x-values show much closer agreement to the empirical q_x-values.

Let us now apply the standard interpolation technique as proposed by Elandt–Johnson and Johnson (1980) to the empirical abridged $l(x)$-values for the same populations and the same period as before. Using (2.1), we calculated a complete set of $l(x)$-values for each sex. Then, we transformed these interpolated values of $l(x)$ to probabilities of dying by the use of the formula $q_x = 1 - l(x + 1)/l(x)$.

The results, denoted $\hat{q}_x^{(L)}$, L for Lagrange, are presented in column 4 of Tables A1 and A2, for males and females respectively. Comparing these values with those obtained by our main technique, we observe that the performance of our technique is superior at the beginning of the age interval, and for the young adult ages, while for the later adult ages the Lagrange interpolation gives somewhat better results. This is natural because

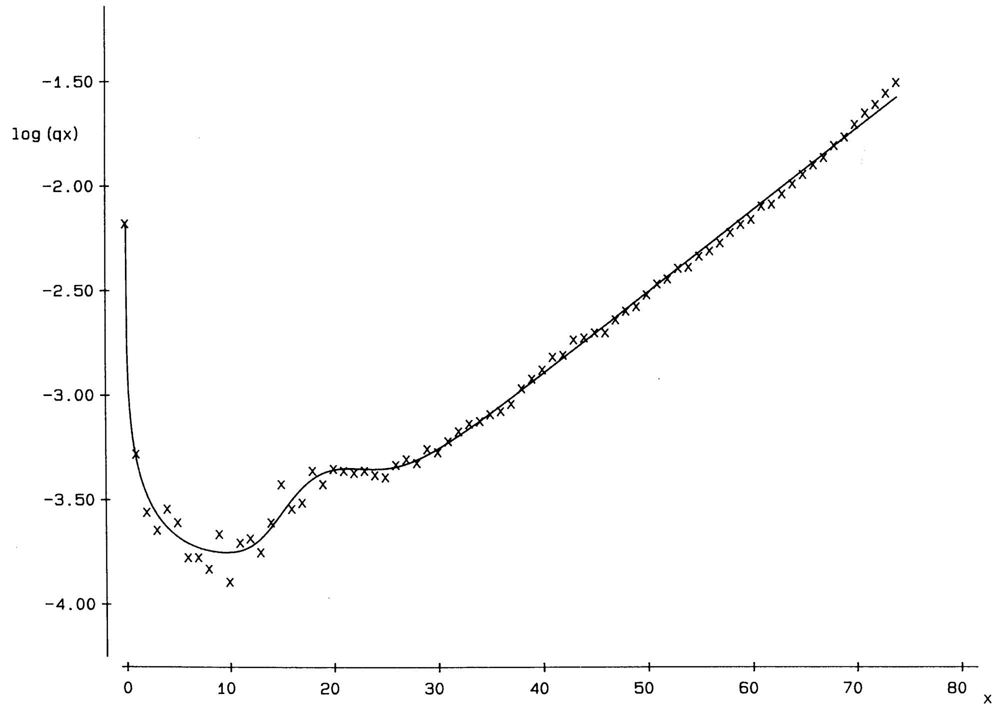

Fig. 2. Empirical q_x-values for Swedish females, 1976–80 (crosses) and computed q_x-values, main expansion technique (solid line)

in our technique we estimate the parameters of the H&P formula through minimization of (4.4). This sum can also be regarded as weighted with weights equal to $1/{}_nq_x^2$. Thus, minimizing (4.4) the algorithm takes mostly into consideration the younger ages, where the weights are heavier as they correspond to smaller ${}_nq_x$-values. However, the performance of our technique after the additional adjustment is much superior at the younger ages and equally good at the later ages.

Now we turn to Reed's procedure. The probabilities of dying, q_{x+2} ($x = 5, 10, \ldots$) were calculated by means of the approximative formula (2.2) using the empirical five-year mortality rates for the same populations and the same periods as before. Reed's procedure was performed, as described before, for both males and females. The resulting probabilities of dying, $\hat{q}_x^{(R)}$, R for Reed, are displayed in Tables A1 and A2 (column 5), for males and females respectively. Comparing these values with those obtained using our main expansion technique we can easily observe that our technique gives much better results for both populations.

One natural criterion for judging the appropriateness of our expansion technique is the sum of squares of the relative deviations between the empirical and the calculated q_x-values

$$\sum_x \left(\frac{\hat{q}_x}{q_x} - 1 \right)^2 \tag{5.1}$$

where $\hat{q}_x$ is the calculated quantity, and q_x is the corresponding empirical one. As an alternative criterion one can also use

$$\sum_x \frac{E_x}{q_x(1 - q_x)} (\hat{q}_x - q_x)^2 \tag{5.2}$$

where E_x is the number of persons exposed to risk at exact age x, the quantities

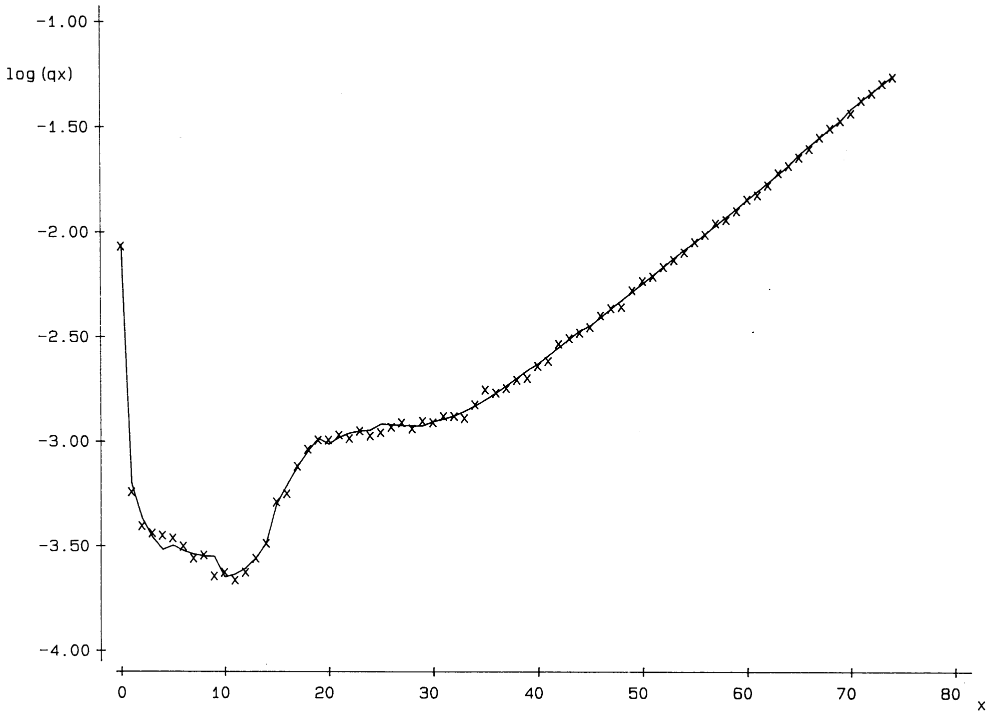

Fig. 3. Empirical q_x-values for Swedish males, 1976–80 (crosses) and computed q_x-values, after additional adjustment (solid line)

$E_x/(q_x(1 - q_x))$ are equal to the reciprocals the binomial distribution variances of the empirical q_x, while q_x, and $\hat{q}_x$ are as before. The reason for using (5.2) as an alternative criterion is that the weighted sum in question takes into consideration mostly those residuals which correspond to q_x-values with low variances, i.e., to those empirical measures which are least affected by stochastic fluctuations. The values of the two goodness-of-fit criteria are given in Table 5.2. It is easily seen that the additional adjustment gives better results than the other three methods. We shall discuss these comparisons in more detail.

The values of (5.1) for the Lagrange interpolation are equal to 0.969 and 0.876 for males and females, respectively. These values

Table 5.2. Goodness-of-fit criteria

	Values of (5.1)				Values of (5.2)			
	Males		Females		Males		Females	
for x equal to	0–74	5–74	0–74	5–74	0–74	5–74	0–74	5–74
Main expansion technique	0.410	0.370	0.795	0.684	206.9	202.2	521.6	513.5
Additional adjustment	0.197		0.591		82.8		99.7	
Lagrange interpolation	0.969		0.876		147.1		86.9	
Reed's technique		1.540		1.928		372.2		545.6

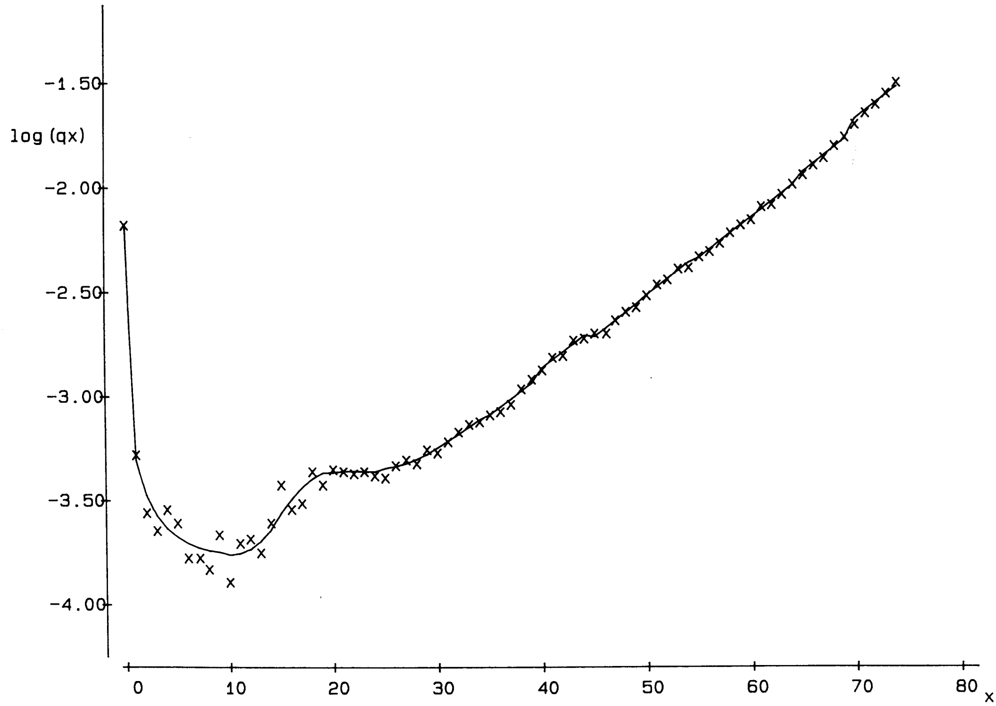

Fig. 4. Empirical q_x-values for Swedish females, 1976–80 (crosses) and computed q_x-values, after additional adjustment (solid line)

are somewhat higher than the corresponding values obtained using our main expansion technique (0.410 for males and 0.795 for females) and much higher than those obtained after the additional adjustment being equal to 0.197 and 0.591 for males and females, respectively. The values of (5.2) for our main expansion technique (206.9 for males and 521.6 for females) are higher than the corresponding values for the Lagrange interpolation for both sexes. However the values of (5.2) for the resulting q_x-values of our technique after the additional adjustment have been essentially reduced, being equal to 82.8 for males and 99.7 for females and thus much lower than the corresponding value (147.1) for Lagrange interpolation for males and only somewhat higher than the corresponding value (86.9) for Lagrange interpolation for females. Turning now to Reed's technique, the values of (5.1), for $x \geqslant 5$, are equal to 1.540 for males and 1.928 for females. These values should be compared with the values of (5.1) for our technique, for $x \geqslant 5$, which are much lower, being 0.370 and 0.684 for males and females, respectively. The values of (5.2) for $x \geqslant 5$ are also lower for our expansion technique, being equal to 202.2 for males and 513.5 for females, than the corresponding values for Reed's technique which are equal to 372.2 and 545.6 for males and females respectively. These results speak for the superiority of our technique.

6. Some Concluding Remarks

In this paper we have outlined and demonstrated a technique for expanding an abridged life table. This technique provides a new way to expand a life table through direct estimation of the complete set of probabilities of dying, q_x, having as starting point the abridged $_nq_x$-values. This is done in a controlled and specified way using a formula which efficiently describes the age pattern of mortality. This might be the main advantage of our procedure in comparison with conventional interpolation formulae applied to tabulated $l(x)$-values, formulae which do not take advantage of the fact that the function to be interpolated is of a special type, viz., a survivor function.

The additional adjustment applied to the results of our main expansion technique has proved very efficient. The resulting q_x-values exhibit a much closer agreement with the empirical ones than the results of the main expansion technique. Nevertheless we should also mention that the adequacy of this additional adjustment is in some way dependent on the quality of the empirical set of $_nq_x$-values. If this empirical set is incomplete or unstable, the q_x-values resulting from the main expansion technique might be more accurate in the sense that they are free from irregular fluctuations which affect the empirical measures and therefore more precise and realistic, the real mortality underlying the observations being a smooth curve.

A technical point that is worth emphasizing is that the procedure can easily be carried out by the use of a well-known NAG algorithm.

Appendix

Table A1. Swedish mortality data, q_x-values, males, 1976–80, multiplied by 100,000

x	(1) q_x	(2) $\hat{q}_x$	(3) $\hat{q}'_x$	(4) $\hat{q}_x^{(L)}$	(5) $\hat{q}_x^{(R)}$	x	(1) q_x	(2) $\hat{q}_x$	(3) $\hat{q}'_x$	(4) $\hat{q}_x^{(L)}$	(5) $\hat{q}_x^{(R)}$
0	869	869	869			40	232	230	235	239	234
1	58	64	63	37		41	245	252	257	260	250
2	40	44	43	45		42	296	276	282	283	268
3	37	36	35	47		43	315	302	309	308	290
4	36	31	30	45		44	335	331	339	335	314
5	35	28	32	41	64	45	356	364	358	362	341
6	32	27	30	35	43	46	405	400	393	395	372
7	28	26	29	29	29	47	438	439	432	432	407
8	29	25	28	24	21	48	445	482	474	473	447
9	23	25	28	20	17	49	535	530	521	520	492
10	24	25	22	12	17	50	594	582	571	573	542
11	22	26	23	17	20	51	623	639	627	629	598
12	24	28	25	25	26	52	693	702	689	690	661
13	28	31	28	34	34	53	749	771	757	756	731
14	33	37	33	45	44	54	816	848	831	829	810
15	52	45	50	57	54	55	915	931	904	906	897
16	57	56	62	68	65	56	994	1023	993	994	993
17	77	68	76	78	76	57	1128	1123	1091	1091	1099
18	93	81	90	87	86	58	1172	1234	1198	1197	1217
19	103	93	104	95	94	59	1294	1356	1316	1315	1346
20	103	104	97	100	101	60	1473	1489	1452	1442	1488
21	109	112	105	105	102	61	1544	1635	1595	1587	1644
22	105	117	110	110	103	62	1721	1796	1751	1749	1815
23	114	120	113	113	105	63	1967	1971	1923	1930	2001
24	108	121	114	116	108	64	2129	2164	2111	2130	2205
25	112	121	122	117	116	65	2331	2375	2383	2345	2427
26	119	120	121	119	123	66	2559	2606	2615	2593	2669
27	125	119	120	120	129	67	2901	2860	2869	2865	2933
28	117	118	119	122	134	68	3203	3136	3147	3166	3218
29	127	118	119	124	140	69	3470	3439	3451	3498	3528
30	125	120	125	125	146	70	3782	3770	3927	3866	3864
31	134	123	128	129	152	71	4348	4143	4304	4267	4227
32	134	128	133	134	158	72	4714	4525	4715	4706	4620
33	131	134	140	141	164	73	5245	4955	5163	5186	5044
34	152	143	149	149	171	74	5646	5424	5651	5712	5502
35	179	153	159	160	179						
36	173	164	171	172	187						
37	182	178	185	186	197						
38	199	193	201	201	207						
39	203	211	219	219	220						

Table A2. Swedish mortality data, q_x-values, females, 1976–80, multiplied by 100,000

x	(1) q_x	(2) $\hat{q}_x$	(3) $\hat{q}'_x$	(4) $\hat{q}_x^{(L)}$	(5) $\hat{q}_x^{(R)}$	x	(1) q_x	(2) $\hat{q}_x$	(3) $\hat{q}'_x$	(4) $\hat{q}_x^{(L)}$	(5) $\hat{q}_x^{(R)}$
0	668	668	668			40	134	125	136	139	129
1	53	49	49	39		41	154	138	149	152	136
2	28	33	33	35		42	157	151	163	165	145
3	23	27	27	31		43	186	165	178	178	154
4	29	23	23	28		44	190	181	194	190	165
5	25	21	21	24	30	45	201	197	192	196	177
6	17	20	20	21	23	46	201	216	210	211	191
7	17	19	19	19	19	47	232	236	230	229	207
8	15	18	18	17	16	48	255	258	251	250	226
9	22	18	18	16	15	49	268	282	275	274	247
10	13	18	17	11	16	50	305	309	304	306	270
11	20	18	18	15	17	51	344	338	333	335	297
12	21	19	19	20	19	52	363	370	364	366	327
13	18	21	20	25	22	53	409	405	398	398	361
14	25	23	23	29	26	54	414	443	436	432	400
15	38	27	28	31	29	55	467	484	459	467	443
16	29	32	32	34	33	56	495	530	503	507	491
17	31	36	37	37	36	57	541	580	550	550	544
18	44	40	40	40	39	58	607	635	602	599	604
19	38	42	43	42	42	59	662	695	659	651	670
20	45	44	43	42	43	60	701	760	714	711	743
21	44	45	44	43	39	61	811	832	781	778	824
22	43	45	44	44	38	62	829	910	855	853	913
23	44	44	44	45	39	63	928	996	935	937	1011
24	42	44	44	46	43	64	1038	1089	1023	1032	1119
25	41	45	45	46	50	65	1150	1192	1182	1135	1238
26	47	45	46	47	56	66	1280	1304	1293	1260	1367
27	50	47	48	48	62	67	1385	1426	1414	1404	1509
28	48	49	50	51	68	68	1579	1560	1547	1570	1665
29	56	52	53	54	73	69	1733	1706	1691	1760	1834
30	54	55	56	59	78	70	1995	1865	2114	1957	2018
31	61	60	61	62	83	71	2262	2039	2311	2209	2218
32	68	64	66	66	87	72	2486	2229	2525	2502	2436
33	74	70	71	71	92	73	2812	2436	2760	2837	2672
34	76	76	77	76	96	74	3168	2662	3015	3219	2928
35	82	82	82	80	101						
36	85	89	89	88	106						
37	92	97	97	97	111						
38	109	106	106	108	116						
39	121	116	115	120	122						

7. References

Elandt–Johnson, R. and Johnson, N. (1980). Survival Models and Data Analysis. New York: John Wiley.

Gill, P. and Murray, W. (1978). Algorithms for the Solution of the Nonlinear Least-Squares Problem. Journal of Numerical Analysis, SIAM, 15, 977–992.

Hartmann, M. (1987). Past and Recent Attempts to Model Mortality at All Ages. Journal of Official Statistics, 3, 19–36.

Heligman, L. and Pollard, J.H. (1980). The Age Pattern of Mortality. Journal of the Institute of Actuaries, 107, 49–80.

Kostaki, A. (1985). A Comparison of the Age Patterns of Mortality in Sweden and Greece. Report, Coden: Lusadgd/SAST-3107/1-13, Department of Statistics, University of Lund, Sweden.

Kostaki, A. (1987). The Heligman–Pollard Formula as a Technique for Expanding an Abridged Life Table. Report, Coden: Lusadgd/SAST-3116/1-15, Department of Statistics, University of Lund, Sweden.

Mode, C. and Busby, R. (1982). An Eight Parameter Model of Human Mortality–the Single Secrement Case. Bulletin of Mathematical Biology, 44, 647–659.

Statistics Sweden (1984). Livslängdstabeller för årtiondet, 1971–80. Stockholm: Statistics Sweden.

United Nations (1988a). MORTPAK. The United Nations Software Package for Mortality Measurement (Batch-oriented Software for the Mainframe Computer). New York: United Nation, 114–118.

United Nations (1988b). MORTPAK-LITE. The United Nations Software Package for Mortality Measurement (Interactive Software for the IBM-PC and Compatibles). New York: United Nation, 111–114.

Valaoras, V. (1984). The 1980 Life Tables for Greece. Publications of the Academy of Athens, 59, 405–436.

Received June 1989
Revised April 1991