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Unit Root Properties of Seasonal Adjustment and Related Filters

William R. Bell[1]

Linear filters used in seasonal adjustment (model-based or from the X-11 method) contain unit root factors in the form of differencing operators and seasonal summation operators. The extent to which the various filters (seasonal, seasonal adjustment, trend, and irregular) contain these unit root factors determines whether the filters reproduce or annihilate (*i*) polynomial functions of time, and (*ii*) fixed seasonal effects. This article catalogs which unit root factors are contained by the various filters for the most common approaches to model-based seasonal adjustment, and for X-11 seasonal adjustment with or without forecast extension. Both symmetric and asymmetric filters are considered.

Key words: Time series; ARIMA model; X-11 seasonal adjustment; trend estimation.

1. Introduction

Common approaches to model-based seasonal adjustment are based on linear filters. The same is true of the widely used X-11 method of seasonal adjustment in either the additive or log-additive modes (Ladiray and Quenneville 2001), and Young (1968) asserted this can be regarded as approximately true for X-11's multiplicative mode. These linear filters contain unit root factors in two forms that are of interest here. One is differencing operators, $(1 - B)^d$ for some integer $d \geq 1$, where B is the backshift operator ($By_t = y_{t-1}$). The other is the seasonal summation operator, denoted here as $U(B) = 1 + B + \ldots + B^{s-1}$, where s is the seasonal period.

Interest in the presence of unit root factors in filters stems from the fact that this determines whether given filters annihilate or reproduce polynomial functions of time (e.g., $\alpha_0 + \alpha_1 t$) or fixed seasonal effects. For polynomial functions of time, suppose $\omega_s(B) = \sum_j \omega_{sj} B^j$ is a seasonal filter and $\omega_N(B) = 1 - \omega_s(B)$ is the complementary seasonal adjustment filter. If, for example, $\omega_s(B)$ contains $(1 - B)(1 - F) = -F(1 - B)^2$, where $F = B^{-1}$ is the forward shift operator, then $\omega_s(B)$ annihilates a linear time trend (since $(1 - B)^2[\alpha_0 + \alpha_1 t] = 0$), while $\omega_N(B)$ reproduces this function. More generally, if $\omega_s(B)$ contains $(1 - B)^d$ for $d > 0$, then it will annihilate polynomials of degree up to $d - 1$, and $\omega_N(B)$ will reproduce them. Sections 3–6 show that the various seasonal filters

[1] U.S. Census Bureau, Washington, DC 20233, U.S.A. Email: William.R.Bell@census.gov
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considered here all contain a $(1 - B)^d$ factor, though of varying degrees d. This is also true of the various irregular filters.

Fixed seasonal effects constitute a deterministic pattern that repeats itself every year and also sums to 0 over any consecutive 12 months (or four quarters) of data. These patterns can be expressed as regression effects, $\sum_i \beta_i x_{it}$, where the x_{it} are either seasonal contrast variables or, equivalently, trigonometric terms at the seasonal frequencies. (See Findley et al. 1998 or U.S. Census Bureau 2009 for specifics.) For fixed seasonal effects the relevant question is whether a seasonal adjustment filter $\omega_N(B)$ annihilates these effects, so that the corresponding seasonal filter $\omega_S(B)$ reproduces them. This will be the case if $\omega_N(B)$ contains $U(B)$. Section 8 notes that indeed all the seasonal adjustment filters considered here, along with the corresponding trend and irregular filters, include a factor $U(B)$. Symmetric seasonal adjustment, trend, and irregular filters contain not just $U(B)$, however, but $U(B)U(F) = U(B)U(B)F^{11}$ for model-based, and $U(B)(1 + F)$ for X-11. (Note that $1 + F$ is a factor of $U(F)$.)

This article catalogs results on unit root factors in seasonal, seasonal adjustment, trend, and irregular filters used by various proposed model-based approaches to seasonal adjustment, and in such filters used by the X-11 method. Both symmetric and asymmetric filters are considered. Some of the results have no doubt been noted before for specific cases, and the results for model-based filters are obvious from their formulas. The general results given here for X-11 filters appear to be new. The focus, though, is not so much on deriving new results, but rather on collecting and discussing the complete set of results for all these filters.

Section 2 reviews the general form of seasonal time series models that have been used both for developing model-based seasonal adjustment methods and for forecast extension in the X-11-ARIMA (Dagum 1980) and X-12-ARIMA programs. The key feature of these models that is of interest here is their use of nonseasonal and seasonal differencing operators, yielding differencing of the observed series by $(1 - B)^{d-1}(1 - B^s) = (1 - B)^d U(B)$ for some $d > 0$. Section 2 also provides two theorems on reproduction of deterministic functions (polynomials and fixed seasonal effects) when forecasting from such models. These results are used in the subsequent sections to obtain results on unit root properties of asymmetric filters.

Sections 3–6 present the results on nonseasonal unit root factors $(1 - B)$ in the various seasonal and irregular filters. Section 3 considers model-based filters, Section 4 X-11 symmetric filters, Section 5 asymmetric filters obtained by applying X-11 symmetric filters to a series with full forecast and backcast extension, and Section 6 the original X-11 asymmetric filters. Section 7 then provides an illustration of the results of Section 6. Section 8 presents and discusses results on seasonal unit root factors in model-based and X-11 filters, both symmetric and asymmetric. Finally, Section 9 provides a closing discussion.

In the presentation we usually assume the time series is monthly ($s = 12$). Corresponding results for quarterly series are generally either the same as those for monthly series, or follow from the latter with obvious modifications (e.g., replace 12 by 4). The modifications needed for quarterly series will be noted. To limit the length of the presentation, proofs and derivations of results are omitted, but can be found in Bell (2010).

2. Time Series Models Used in Seasonal Adjustment

The additive decomposition used for seasonal adjustment is

$$y_t = S_t + T_t + I_t \tag{1}$$

where y_t is the observed time series (possibly after transformation, e.g., taking logarithms), and S_t, T_t, and I_t are the seasonal, trend, and irregular components. We also let $N_t = T_t + I_t = y_t - S_t$ denote the nonseasonal component, the estimate of which is known as the seasonally adjusted series. Most of the models that have been proposed for model-based seasonal adjustment use component models that can be written in the following form:

$$\begin{aligned} U(B)S_t &= u_t \\ (1-B)^d T_t &= v_t \\ I_t &\sim i.i.d.\, N(0, \sigma_I^2) \end{aligned} \tag{2}$$

where u_t and v_t have mean zero for all t and are independent of each other and of I_t. Typically, u_t and v_t follow stationary autoregressive moving average (ARMA) Gaussian models (Box and Jenkins 1970), though particulars of the models for u_t and v_t are not needed here for the most part. We require only that u_t and v_t be stationary with autocovariance functions $\gamma_u(k) = \mathrm{Cov}(u_t, u_{t+k})$ and $\gamma_v(k) = \mathrm{Cov}(v_t, v_{t+k})$ that are absolutely summable, that is, $\sum_{k=-\infty}^{\infty}|\gamma_u(k)| < \infty$ and $\sum_{k=-\infty}^{\infty}|\gamma_u(k)| < \infty$. This summability condition is satisfied by stationary ARMA models. We let $\gamma_u(B) = \sum_{k=-\infty}^{\infty}\gamma_u(k)B^k$ and $\gamma_v(B) = \sum_{k=-\infty}^{\infty}\gamma_v(k)B^k$ denote the autocovariance generating functions (ACGFs) of u_t and v_t. The ACGF of I_t is just σ_I^2.

The Gaussian assumption made above is not essential. Without it, forecasting and signal extraction results given later are interpretable as linear projections, though not as conditional expectations. Also, we could extend (2) to let I_t follow a stationary and invertible ARMA model instead of requiring it to be white noise (independent and identically distributed over t). This extension would not materially alter the results presented here.

With the component models given in (2), we can write the model for y_t in terms of the component model for the differenced series $w_t = (1-B)^d U(B) y_t = (1-B)^{d-1}(1-B^{12})y_t$:

$$w_t = (1-B)^d U(B) y_t = (1-B)^d u_t + U(B)v_t + (1-B)^d U(B) I_t. \tag{3}$$

The ACGF of w_t is given by

$$\gamma_w(B) = (1-B)^d(1-F)^d\gamma_u(B) + U(B)U(F)\gamma_v(B) + (1-B)^d U(B)(1-F)^d U(F)\sigma_I^2.$$

We shall assume that the models for w_t and y_t given by (3) are invertible.

The model framework of Equations (1)–(3) covers the canonical ARIMA (autoregressive integrated moving average) model-based approach to seasonal adjustment as developed by Hillmer and Tiao (1982) and Burman (1980), and implemented in the TRAMO-SEATS software of Gomez and Maravall (1997). This is the most common model-based procedure actually used for official seasonal adjustments. The model framework of (1)–(3) also covers the structural components models of Harvey (1989) and

Durbin and Koopman (2001). The ARIMA models used for forecast extension in the X-11-ARIMA and X-12-ARIMA programs are also of the general form of (3). Regression terms are often added to these models to account for trading-day and other effects. As regression effects do not usually affect the unit root properties of the seasonal adjustment methods examined here, we shall not, with one exception, bother including them in the models we present here. The one exception involves trend constants—an overall nonzero mean for the differenced series, w_t. Trend constants do affect the degree of polynomials annihilated and reproduced by some of the various filters, so these are explicitly considered.

We now establish two results on forecasting with models of the general form given by (3). Let $\delta(B) = 1 - \delta_1 B - \cdots - \delta_{d+11} B^{d+11} = (1 - B)^{d-1}(1 - B^{12})$. Bell (2004, Section 12.3.3) notes that minimum mean squared error (MMSE) forecasts $\hat{y}_{t|n}$ of y_t for $t > n$ from finite data $\mathbf{y} = (y_1, \ldots, y_n)'$ satisfy

$$\hat{y}_{t|n} = \delta_1 \hat{y}_{t-1|n} + \cdots + \delta_{d+11} \hat{y}_{t-d-11|n} + \hat{w}_{t|n} \quad (4)$$

where $\hat{y}_{j|n} = y_j$, for $j = 1, \ldots, n$, and $\hat{w}_{t|n} = E(w_t|\mathbf{w})$ is the MMSE forecast of w_t under the Model (3) given the observed differenced data, $\mathbf{w} = (w_{d+12}, \ldots, w_n)'$. Result (4) follows under Assumption A of Bell (1984) about starting values for the series y_t, and is consistent with the standard approach to forecasting nonstationary time series used, for example, in Box and Jenkins (1970). Theorem 1 establishes what happens if the forecast procedure defined by (4) is applied to any deterministic function ξ_t that is annihilated by $\delta(B)$. Note that in doing this the Model (3) is taken as given, so the forecast procedure does not involve fitting (3) to ξ_t taken as data.

Theorem 1: Forecasting via (4) with the model given by (3) reproduces any deterministic function ξ_t that is annihilated by $\delta(B)$. For $\delta(B) = (1 - B)^d U(B)$, these ξ_t include (*a*) polynomials in *t* of degree less than *d*, (*b*) fixed seasonal effects, and (*c*) linear combinations of these two. Higher-order deterministic functions, such as polynomials of degree *d* or more, are not reproduced.

Since the backward model for y_t has the same form as the usual (forward) model, just with *F* replacing *B* in the AR, differencing, and MA operators (Box and Jenkins 1970, pp. 197–198), the results of the theorem also hold for MMSE backcasting.

Models of the general form (3) can be extended by addition of a trend constant, which is a nonzero mean μ_w for the differenced series w_t:

$$(1 - B)^d U(B) y_t \equiv w_t = \mu_w + \tilde{w}_t \quad (5)$$

where $\tilde{w}_t = w_t - \mu_w$ has $E(\tilde{w}_t) = 0$, and now $\tilde{w}_t$ follows the model given for w_t in (3). To obtain forecasts from this model, we modify (4) to

$$\hat{y}_{t|n} = \mu_w + \delta_1 \hat{y}_{t-1|n} + \cdots + \delta_{d+11} \hat{y}_{t-d-11|n} + \tilde{w}_{t|n} \quad (6)$$

where $\tilde{w}_{t|n} = E(\tilde{w}_t|\mathbf{w})$. We also need to substitute an estimate of μ_w into (6). We could use the sample mean, $\bar{w}$, of w_t, or a generalized least squares (GLS) estimate

$$\hat{\mu}_w = (\mathbf{1}'\Sigma^{-1}\mathbf{w})/(\mathbf{1}'\Sigma^{-1}\mathbf{1}) \quad (7)$$

where $\mathbf{1}' = (1, \ldots, 1)$ and Σ is any positive definite covariance matrix. Setting $\Sigma \propto I$ gives the least-squares estimate, $\bar{w}$. Setting $\Sigma = \Sigma_w$ gives the optimal GLS estimate under the Model (5).

For the Model (5), we have the following analog to Theorem 1. Note that the model defined by (5) and (3) is taken as given except for the estimation of μ_w by (7), which is regarded as part of the forecast procedure. The use of (7) is for finite data. With data extending into the infinite past, $\hat{\mu}_w$ can be assumed to converge to μ_w, which can then be taken as known.

Theorem 2: Forecasting via (6) with the trend constant Model (5) reproduces (*a*) polynomials in t up to degree d, (*b*) fixed seasonal effects, and (*c*) linear combinations of these two.

Theorem 2 can be extended to a model where $E(w_t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_h t^h$ to show that forecasting with this model reproduces polynomials up to degree $h + d$. The simplest version of this would have $h = d = 1$, which implies a quadratic time trend in the data. As use of quadratic or higher-order polynomial trends should be very unusual in practice, we shall not pursue this additional generality here.

3. Differencing Factors in Model-Based Filters

Results on unit root factors in model-based filters follow directly from expressions for the signal extraction estimates that form the basis of model-based seasonal adjustment. We consider the results for three cases defined by the amount of data used in the signal extraction: the doubly infinite realization $\{y_t \text{ for } t = -\infty, \ldots, \infty\}$; the semi-infinite realization $\{y_t \text{ for } t \leq n\}$ for some finite n; and the finite vector of observations $\mathbf{y} = (y_1, \ldots, y_n)'$. The first of these leads to symmetric infinite filters, the second to asymmetric infinite filters, and the third to finite filters (of which at most one will be symmetric).

Bell (1984) presents results on signal extraction with a doubly infinite realization of y_t for models of the form of (l)–(3). The MMSE linear signal extraction estimate of S_t given $\{y_t \text{ for } t = -\infty, \ldots, \infty\}$ is $\hat{S}_t = w_S(B) y_t$ where

$$\omega_S(B) = \frac{\gamma_u(B)}{\gamma_w(B)}(1-B)^d(1-F)^d. \tag{8}$$

Analogous to (8), the linear filters for the MMSE estimates of N_t, T_t, and I_t are

$$\omega_N(B) = \frac{\gamma_z(B)}{\gamma_w(B)} U(B)U(F) \tag{9}$$

$$\omega_T(B) = \frac{\gamma_v(B)}{\gamma_w(B)} U(B)U(F) \tag{10}$$

$$\omega_I(B) = \frac{\sigma_I^2}{\gamma_w(B)} U(B)U(F)(1-B)^d(1-F)^d. \tag{11}$$

In (9), $\gamma_z(B) = \gamma_v(B) + (1-B)^d(1-F)^d\sigma_I^2$ is the ACGF of $z_t \equiv (1-B)^d N_t = v_t + (1-B)^d I_t$. It can be shown that $\omega_N(B) = 1 - \omega_S(B) = \omega_T(B) + \omega_I(B)$. In what follows, we shall use the notations $\omega_S(B)$, $\omega_N(B)$, $\omega_T(B)$, and $\omega_I(B)$ not just for the formulas in (8)–(11), but generically for the model-based seasonal, seasonal adjustment, trend, and irregular filters for all three cases—the symmetric, semi-infinite, and finite. Which case applies will be noted in the text.

Table 1 summarizes the results for the $(1 - B)$ factors appearing in the MMSE model-based seasonal and irregular filters. The corresponding seasonal adjustment and trend filters reproduce polynomials up to the degrees shown in the table. Results for the symmetric case follow directly from Equations (8) and (11), for the semi-infinite case from analogous filter expressions given by Bell and Martin (2004), and for the finite case from matrix expressions given by McElroy (2009). Bell (2010) also gives these filter expressions.

The most common models used in model-based seasonal adjustment have $d = 2$. In this case, we see from Table 1 that the symmetric seasonal adjustment filters will reproduce cubic polynomials of t, while the asymmetric and finite seasonal adjustment filters will reproduce only linear polynomials. Less commonly used models have $d = 1$, and for these the symmetric seasonal adjustment filters will reproduce only linear functions of time, while the asymmetric and finite seasonal adjustment filters reproduce only constants. The same remarks apply to the trend filters. Note that values of d other than 1 and 2 are extremely uncommon in practice.

Notice from (8) and (11) that the symmetric $\omega_S(B)$ and $\omega_I(B)$ contain not just the $(1 - B)^d$ needed to remove the nonstationarities in the trend component; they contain $(1 - B)^d(1 - F)^d$. Similarly, from (9)–(11), the symmetric $\omega_N(B)$, $\omega_T(B)$, and $\omega_I(B)$ contain not just the $U(B)$ needed to remove the seasonal nonstationarities of S_t, they contain $U(B)U(F)$. In the context of time series modeling, application of more differences than needed to render a series stationary is termed "overdifferencing" (Harvey 1981). The overdifferenced series follows a model that includes $(1 - B)^k$ as an MA polynomial, where k is the excessive number of differences applied. This overdifferencing by symmetric filters is worth noting if one considers time series modeling of estimated components. Such modeling faces other issues, however, including nonstationarities induced by end effects from the different asymmetric filters applied at different time points. (See Bell 1995 for related discussion.) Section 8 discusses the implications of the symmetric $\omega_N(B)$, $\omega_T(B)$, and $\omega_I(B)$ containing $U(B)U(F)$.

The asymmetric and finite seasonal filters include only $(1 - B)^d$, and so do not overdifference. This is also true of the asymmetric and finite irregular filters.

Something not clear from (8)–(11) is whether these filters contain additional unit root factors beyond those obvious from inspection. Bell (2010) notes that $\omega_I(B)$ will not include additional unit root factors, while for $\omega_S(B)$, $\omega_N(B)$, and $\omega_T(B)$, additional unit root factors are possible if they appear in the MA polynomials of the ARIMA models for S_t, N_t, or T_t. For example, Hillmer and Tiao (1982, p. 67) examine a model for which the canonical trend component has a factor of $(1 + B)$ in its MA polynomial. While potential

Table 1. Differencing factors in model-based seasonal and irregular filters

Amount of data used	Differencing $(1 - B)$ factors	Annihilates
Symmetric (doubly infinite)	$(1 - B)^d(1 - F)^d$	polynomials up to degree $2d - 1$
Asymmetric (semi-infinite) or finite	$(1 - B)^d$	polynomials up to degree $d - 1$

additional unit root factors in the filters considered can obviously be examined for any particular model, general results are difficult to give.

If the model with a trend constant, (5), is used, we would revise (2) to $(1-B)^d T_t \equiv v_t = \mu_v + \tilde{v}_t$, and note from (3) that $\mu_w = E(w_t) = U(B)E(v_t) = 12\mu_v$ since $E(u_t) = E(I_t) = 0$. Let $\eta_t = [\mu_w/12(d!)]t^d$. We can think of $E(y_t) = \eta_t$ since then $U(B)(1-B)^d E(y_t) = \mu_w$. (We could add lower-order polynomial terms or fixed seasonal effects to η_t, but these would be differenced to zero in (3), and so would not be estimable.) In the finite sample case we substitute $\hat{\mu}_w$ for μ_w. Model-based signal extraction to estimate S_t and I_t then applies $\omega_S(B)$ and $\omega_I(B)$ to $y_t - \eta_t$. For asymmetric or finite sample signal extraction, this can be shown to annihilate polynomials up to degree d, not just the degree $d-1$ obtained for models without trend constants. (See Bell 2010 for more details.) Corresponding signal extraction estimation of N_t and T_t then reproduces polynomials up to degree d. Note that the symmetric signal extraction filters (8) and (11) can be applied directly to y_t rather than to $y_t - \eta_t$, since these symmetric filters annihilate polynomials of degree $2d-1 \geq d$.

Two alternative approaches to doing finite sample signal extraction calculations are worth mentioning. First, if the models can be put in state-space form (as can ARIMA component models), signal extraction can be done using the Kalman filter together with a suitable initialization and a smoothing algorithm. See Bell and Hillmer (1991) or Durbin and Koopman (2001) for this approach. Second, Cleveland (1972) suggested extending the series as necessary with MMSE forecasts and backcasts so the MMSE symmetric filters can be applied. (This approach also applies to the semi-infinite case with only forecast extension needed.) The infinite filter weights decay sufficiently fast for this procedure to converge, so that in practice only a finite, though possibly large, number of forecasts and backcasts are needed. For ARIMA model-based seasonal adjustment, G. Tunnicliffe-Wilson suggested an algorithm (reported in Burman 1980) implementing this approach via a reduced set of calculations. Both these approaches produce signal extraction estimates identical to those from the matrix formulas of McElroy (2009), and hence all have the same unit root properties.

The approach of Cleveland provides an instructive way to achieve the results of Table 1 for semi-infinite and finite filters. Forecast extension with Model (3) reproduces polynomials up to degree $d-1$, while the subsequent application of the symmetric versions of $\omega_N(B)$ and $\omega_T(B)$ will reproduce polynomials up to degree $2d-1 \geq d$. Hence, asymmetric (or finite) $\omega_N(B)$ and $\omega_T(B)$ will reproduce polynomials up to degree $d-1$, and asymmetric (or finite) $\omega_S(B)$ and $\omega_I(B)$ annihilate such polynomials. The same rationale also applies to the trend constant model (5), although with this model forecasting, and thus signal extraction estimation of N_t and T_t, reproduces polynomials up to degree d. Note that the limiting factor here is the degree of polynomial reproduced by the forecast extension, since the symmetric seasonal adjustment and trend filters will reproduce polynomials of as high or higher degree. We shall use analogous considerations in Sections 5 and 6 to determine unit root properties of X-11 asymmetric filters obtained with or without forecast extension.

4. Differencing Factors in X-11 Symmetric Filters

Wallis (1974) lists the filtering steps used in X-11 seasonal adjustment with the additive decomposition (1) (and the log-additive decomposition when y_t is a logged series). (See also Ladiray and Quenneville 2001, Section 2.4). Specific X-11 filters are determined by the seasonal and trend MAs used, whether specified directly by the user or chosen automatically by the program. As the particular choices of MAs will not affect the unit root results of interest here, we shall not go into details about the choice of MAs. For this, see Ladiray and Quenneville (2001, Chapter 3).

Bell and Monsell (1992) note that the X-11 filters can be expressed symbolically in terms of their MAs, and so provide the following expression for the X-11 symmetric seasonal filter, which we denote as $\omega_S^{X11}(B)$:

$$\omega_S^{X11}(B) = [1 - \mu(B)]\lambda_2(B)[1 - H(B)\{1 - [1 - \mu(B)]\lambda_1(B)[1 - \mu(B)]\}] \tag{12}$$

where

$$\mu(B) = 2 \times 12 \text{ trend MA} = \frac{1}{24}F^6(1+B)U(B) = \frac{1}{24}(F^6 + 2F^5 + \cdots + 2B^5 + B^6)$$

$$\lambda_1(B) = \text{first seasonal MA, e.g., } \frac{1}{9}(F^{12} + 1 + B^{12})(F^{12} + 1 + B^{12})$$

$$\lambda_2(B) = \text{second seasonal MA, e.g., } \frac{1}{15}(F^{12} + 1 + B^{12})(F^{24} + F^{12} + 1 + B^{12} + B^{24})$$

$$H(B) = \text{Henderson trend MA.}$$

For quarterly series we change 12 to 4 and 24 to 8 in the above expressions, and the $F^6(1 + B)$ to $F^2(1 + B)$ in the definition of $\mu(B)$. The Henderson trend MAs are discussed by Kenny and Durbin (1982), Dagum (1985), and Ladiray and Quenneville (2001, Chapter 3). Given the symmetric seasonal filter $\omega_S^{X11}(B)$, the X-11 symmetric filters for estimating the remaining components are as follows:

$$\omega_N^{X11}(B) = 1 - \omega_S^{X11}(B) \tag{13}$$

$$\omega_T^{X11}(B) = H(B)\omega_N^{X11}(B) \tag{14}$$

$$\omega_I^{X11}(B) = [1 - H(B)]\omega_N^{X11}(B). \tag{15}$$

We shall use the notation $\omega_S^{X11}(B)$, etc., generically to denote X-11 seasonal, seasonal adjustment, trend, and irregular filters also in subsequent sections that cover the case of X-11 asymmetric filters with or without forecast extension.

Unit root properties of the X-11 symmetric filters can be inferred from Expressions (12)–(15) using knowledge of the unit root properties of $\mu(B)$, the seasonal MAs, and the Henderson trend MAs. The latter properties can be determined via numerical zero finding, or by repeated polynomial division by $(1 - B)$ (d times to check for a $(1 - B)^d$ factor), or by $(1 - B^{12})$ or $U(B)$ to check for these factors. This was done and the results are stated as Lemma 1.

Lemma 1: The moving averages used in the X-11 symmetric filters have the following unit root properties for monthly series:

(a) $\mu(B)$, the 2×12 MA, contains $(1 + B)U(B)$
(b) $1 - \mu(B)$ contains $(1 - B)(1 - F)$
(c) $1 - \lambda(B)$ contains $(1 - B^{12})(1 - F^{12}) = (1 - B)(1 - F)U(B)U(F)$ for any of the X-11 seasonal MAs $\lambda(B)$

(d) $1 - H(B)$ contains $(1 - B)^2(1 - F)^2$ for any of the Henderson trend MAs, $H(B)$.

No other factors of $U(B)$, nor additional $(1 - B)$ factors, are contained by these MAs or their complements. For quarterly series change 12 to 4 in (a) and (c).

Note that result (a) follows directly from the definition of $\mu(B)$. Also, since the Henderson trend MAs are explicitly designed to reproduce cubic polynomials (Kenny and Durbin 1982), result (d) must hold.

By manipulating Expressions (12)—(15) and using the results of Lemma 1, we can establish the unit root properties of the X-11 symmetric filters (Bell 2010). The differencing factors contained in the seasonal and irregular filters are listed in Table 2. Comparing Tables 1 and 2, several interesting differences emerge.

First, since it would be very unusual to have a value of d greater than 2 in Model (2), we see that $\omega_S^{X11}(B)$ contains more $1 - B$ factors (effectively 6) than would a model-based symmetric seasonal filter. Consequently, the X-11 symmetric seasonal adjustment filters will reproduce polynomials up to degree 5, while model-based symmetric seasonal adjustment filters will only reproduce polynomials up to degree 3 (if $d = 2$) or 1 (if $d = 1$).

Second, note that $\omega_I^{X11}(B)$ includes $(1 - B)^2(1 - F)^2$, whereas $\omega_S^{X11}(B)$ includes $(1 - B)^3(1 - F)^3$. Though not shown in Table 2, $1-\omega_T^{X11}(B)$ also includes just $(1 - B)^2(1 - F)^2$. This contrasts with the results for model-based filters (Table 1 and the discussion following), where $\omega_s(B)$, $\omega_I(B)$, and $1 - \omega_T(B)$ all include the same $(1 - B)^d(1 - F)^d$ factors. Hence, for X-11, the symmetric trend filter reproduces, and the symmetric irregular filter annihilates, polynomials of lower degree than are annihilated by the symmetric seasonal filter, whereas for model-based filters the degrees of the polynomials annihilated or reproduced by these filters are all the same.

Third, because $\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$ contain as many or more $(1 - B)$ factors as do the corresponding model-based symmetric filters, the remarks of Section 3 about "overdifferencing" by the model-based symmetric filters apply also to $\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$.

Young (1968) provided alternative approximations to X-11 symmetric filters. For the seasonal filter, Young (1968, Eq (4)) omitted two of the steps outlined by Wallis (1974) corresponding to the first two $1 - \mu(B)$ terms in (12). Omitting these terms yields the following approximation, which we denote as $\omega_S^Y(B)$:

$$\omega_S^Y(B) = \lambda_2(B)[1 - H(B)\{1 - \lambda_1(B)[1 - \mu(B)]\}]. \tag{16}$$

Corresponding approximations to the X-11 symmetric seasonal adjustment, trend, and irregular filters start with $\omega_S^Y(B)$ and follow as in (13)—(15). Young argued for considering these filters when applied to logged data as an approximation to X-11's multiplicative decomposition. Though Wallis's (1974) representation of X-11 linear filters is exact, it is so only for additive and log-additive decompositions, and it appears that the question of

Table 2. Differencing factors in X-11 symmetric linear filters

Filter	Differencing $(1 - B)$ factors	Annihilates
$\omega_S^{X11}(B)$	$(1 - B)^3(1 - F)^3$	polynomials up to degree 5
$\omega_I^{X11}(B)$	$(1 - B)^2(1 - F)^2$	polynomials up to degree 3

whether Young's approximation (16), or Wallis's exact version with a log-additive decomposition, provides a better approximation to multiplicative X-11 has not been studied.

Results on the (1-B) factors contained by Young's approximate filters differ somewhat from those given above for Wallis's exact filters. First, $\omega_S^Y(B)$ contains only $(1-B)(1-F)$, so $\omega_S^Y(B)$ annihilates, and $\omega_N^Y(B)$ reproduces, only linear polynomials in t (not the polynomials up to degree 5 of the Wallis representation). Second, $1-\omega_T^Y(B)$ also contains $(1-B)(1-F)$, and while this is less than the $(1-B)^2(1-F)^2$ contained by $1-\omega_T^{X11}(B)$, it is consistent with the result for $\omega_S^Y(B)$, as is the case for model-based symmetric filters. (Note also that the model-based symmetric seasonal and irregular filters for $d=1$ also contain just the factors $(1-B)(1-F)$.) Third, due to the presence of $1-H(B)$ in the analog to Equation (15) for Young's filter, $\omega_I^Y(B)$ includes $(1-B)^2(1-F)^2$, and so annihilates cubic polynomials in t, matching the result in Table 2, but differing from the result for $\omega_S^Y(B)$.

5. Differencing Factors in X-11 Asymmetric Filters with Full Forecast Extension

To deal with the issue of X-11 symmetric filters not being applicable near the ends of time series, Dagum (1975) proposed extending series with forecasts and backcasts from ARIMA models, leading to the X-11-ARIMA method (Dagum 1980). Pierce (1980) and Geweke (1978) pointed out that extending series with optimal (MMSE) forecasts and backcasts, that is, such that, append sufficient forecasts and backcasts to the series so the symmetric filters could be applied at $t=1,\ldots,n$, would minimize mean squared revisions of the seasonally adjusted data. In practice, the true model for a series is unknown, so optimal forecasts cannot be achieved, but the motivating idea behind these articles was that one could find a model good enough that using its forecasts and backcasts could at least reduce the size of the revisions. As noted in Section 2, the programs X-11-ARIMA and X-12-ARIMA use models of the form of (3) or (5). The default option in X-11-ARIMA and X-12-ARIMA is not full forecast extension, however, but rather extension with one year of forecasts. (Backcast extension is usually of less concern. It can be requested in X-12, but the default is no backcast extension.) With this approach, the original X-11 asymmetric filters still play a role. In this section, we consider unit root properties of X-11 filters obtained using full forecast and backcast extension. Section 6 discusses unit root properties of X-11 filters obtained with partial forecast and backcast extension.

Unit root properties of X-11 filters obtained using full forecast extension follow from the results of Table 2, Theorem 1 of Section 2, and the approach described at the end of Section 3. Thus, $(1-B)$ and $U(B)$ factors occur in the various filters according to the lesser of (i) the degree to which they occur in the corresponding X-11 symmetric filter, and (ii) the degree to which polynomials or fixed seasonal effects are reproduced in forecasting by the model used. The results for the differencing factors are given in Table 3. Results on $U(B)$ factors are given in Section 8. As noted in the previous section, we use the generic notation $\omega_S^{X11}(B)$, and so on, here to denote the X-11 asymmetric filters obtained with full forecast extension.

Notice that the results in Table 3 are the same as those in Table 1 for model-based asymmetric filters, with just the noted exceptions that would occur for large values of d that should never occur in practice anyway. This is because the X-11 symmetric filters

Table 3. Differencing factors in X-11 asymmetric linear filters with full forecast extension

Filter	Differencing $(1-B)$ factors*	Annihilates*
$\omega_S^{X11}(B)$	$(1-B)^d$	polynomials up to degree $d-1$
$\omega_I^{X11}(B)$	$(1-B)^d$	polynomials up to degree $d-1$

* For asymmetric $\omega_S^{X11}(B)$ with full forecast extension, the result assumes that $d \leq 6$; for $d > 6$, change d to 6 in the first row. For $\omega_I^{X11}(B)$, the result assumes that $d \leq 4$; for $d > 4$, change d to 4 in the second row.

$\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$ contain as many or more $1-B$ factors (see Table 2) than are ordinarily contained by the model-based symmetric signal extraction filters (Table 1), and so the forecasting results are what limit the number of $(1-B)$ factors in both the asymmetric model-based and asymmetric X-11 filters with full forecast extension.

As was the case for model-based asymmetric filters, the unit root factors shown in Table 3 are just those needed to remove nonstationarities present in the other components according to the Model (2). Thus, in contrast to the results for X-11 symmetric filters, no "overdifferencing" occurs.

The same reasoning applies to Models (5) with trend constants. From Theorem 2 of Section 2, adding the trend constant to the model increases the degree of polynomials reproduced by forecasting by 1. Hence, when full forecast extension uses Model (5), we can increase d to $d+1$ in Table 3. Per the note to Table 3, this assumes that $d \leq 5$ for the first row of the table, and $d \leq 3$ for the last two rows.

Note that the results in Table 3 apply only at those time points for which the symmetric filters cannot be applied and forecast extension is needed. Let the symmetric seasonal filter be written as $\sum_{j=-r}^{r}\omega_{S,j}^{X11}B^j$ with the $2r+1$ symmetric weights $\omega_{S,j}^{X11} = \omega_{S,-j}^{X11}$. We call r the "half-length" of the symmetric filter. Then, the result in Table 3 for $\omega_S^{X11}(B)$ applies for $t = 1, \ldots, r$ and $t = n+1-r, \ldots, n$, while for $t = r+1, \ldots, n-r$, the symmetric seasonal filter is used, so the result given in Table 2 applies. For $\omega_I^{X11}(B)$, the result in Table 3 applies for $t = 1, \ldots, r+p$ and $t = n+1-r-p, \ldots, n$, where p is the half-length of $H(B) = \sum_{j=-p}^{p}H_jB^j$, the symmetric Henderson trend MA. The value of r varies with alternative choices of the seasonal and Henderson trend MAs, a point we discuss further in the next section.

6. Differencing Factors in Original X-11 Asymmetric Filters

To deal with the inapplicability of the symmetric filters except in the middle of sufficiently long time series, the original X-11 program (Shiskin et al. 1967) provided families of asymmetric seasonal and trend MAs used in place of the symmetric versions of $\lambda_1(B)$, $\lambda_2(B)$ and $H(B)$ in Equation (12). Ladiray and Quenneville (2001, Chapter 3) discuss these asymmetric MAs and give their filter weights. There is also need for an "asymmetric version" of $\mu(B)$, as discussed in Bell (2010). The asymmetric seasonal and trend MAs are also needed by the X-11 procedures of X-11-ARIMA and X-12-ARIMA for use when there is no or only partial forecast extension.

The X-11 asymmetric filters carry out the same general sequence of operations as the X-11 symmetric filters (Wallis 1982), and so can still be loosely represented by Expressions (12)–(15). However, the MAs in the asymmetric filters are time-varying, in that when insufficient observations are available to apply a symmetric MA at a given time point, the appropriate asymmetric MA is used. Because of this, we cannot simply expand (12)–(15) as polynomials in B with fixed weights. To determine the unit root properties of the X-11 asymmetric filters, Bell (2010) instead examines what results from applying the filters to sequences ξ_t representing either fixed seasonal effects or polynomial functions of time ($1, t, t^2$, etc.).

The unit root factors in the asymmetric versions of X-11's 3-term, 3×3, 3×5, 3×9, and 3×15 seasonal MAs were found numerically. Filter weights were taken from the X-11 code in the X-12-ARIMA program. Ladiray and Quenneville (2001, p. 45) give weights for the asymmetric 3×3, 3×5, and 3×9 MAs, though, for the 3×9 MAs the weights given are approximations that do not quite preserve the unit root factors of the MAs actually used in the program. Ladiray and Quenneville (2001, pp. 40–44) also provide weights for the asymmetric Henderson trend MAs, and note that these MAs reproduce only constants, not linear functions. Collecting these results gives the following lemma.

Lemma 2: The asymmetric seasonal ($\lambda_t(B)$) and Henderson trend ($H_t(B)$) moving averages used in X-11 have the following unit root properties (for monthly series):

(a) $1 - \lambda_t(B)$ contains $(1 - B^{12}) = (1 - B)U(B)$ for any of the X-11 asymmetric seasonal MAs, $\lambda_t(B)$.
(b) $1 - H_t(B)$ contains $(1 - B)$ for any of the asymmetric Henderson trend MAs, $H_t(B)$.

No other factors of $U(B)$, nor additional $(1 - B)$ factors, are contained by the $1 - \lambda_t(B)$ and $1 - H_t(B)$. For quarterly series change 12 to 4 in (a).

Using Lemma 2, Bell (2010) derives the unit root factors in the original X-11 asymmetric filters (again, denoted generically here as $\omega_S^{X11}(B)$ etc.) These results depend on t, the time point at which the components are being estimated, and on the half-lengths of the seasonal and trend MAs used. Let the half-lengths of the symmetric seasonal MAs, $\lambda_1(B)$ and $\lambda_2(B)$, be m_1 and m_2, respectively. Note that these MAs involve $2m_1 + 1$ and $2m_2 + 1$ weights, including many weights that are zero because the only nonzero weights in the seasonal MAs are at the seasonal lags and leads. Again, let p denote the half-length of the symmetric $H(B)$. The half-length, r, of the full symmetric seasonal filter can then be seen from (12) to be $r = 18 + m_1 + m_2 + p$. As noted in Section 5, the X-11 symmetric seasonal filter applies for $t = r + 1, \ldots, n - r$, and this filter contains $(1 - B)^6$. Table 4 shows how the $(1 - B)$ factors in the original X-11 asymmetric seasonal filters vary across the values of t.

Table 4. Differencing factors in original X-11 asymmetric seasonal filters

Time points	$\omega_S^{X11}(B)$ contains	$\omega_S^{X11}(B)$ annihilates
$t = 1, \ldots, 6 + m_2 + p$ $t = n + 1 - (6 + m_2 + p) \ldots, n$	$(1 - B)$	constants
$t = 7 + m_2 + p, \ldots, r$ $t = n + 1 - r, \ldots, n - (6 + m_2 + p)$	$(1 - B)^3$	polynomials up to degree 2
$t = r + 1, \ldots, n - r$ (symmetric filter)	$(1 - B)^6$	polynomials up to degree 5

We see that, near the ends of the series ($t = 1, \ldots, 6 + m_2 + p$ and $t + n + 1 - (6 + m_2 + p), \ldots, n$), asymmetric $\omega_S^{X11}(B)$ contains only $1 - B$, while for $t = 7 + m_2 + p, \ldots, r$ and $t = n + 1 - r, \ldots, n - (6 + m_2 + p)$, it contains $(1 - B)^3$. The latter provides a transition to the $(1 - B)^6$ contained by the symmetric seasonal filter, which applies for $t = r + 1, \ldots, n - r$. Such a transition does not occur for model-based filters nor for X-11 filters with full forecast extension. In both of these cases, all the asymmetric filters contain the same unit root factors.

For the original X-11 asymmetric trend and irregular filters, we need to extend the first two ranges of time points in Table 4 by p months. Thus, for $t = 1, \ldots, 6 + m_2 + 2p$ and for $t = n + 1 - (6 + m_2 + 2p), \ldots, n$, asymmetric $\omega_T^{X11}(B)$ reproduces and asymmetric $\omega_I^{X11}(B)$ annihilates only constant polynomials. For $t = 7 + m_2 + 2p, \ldots, r + p$ and for $t = n + 1 - (r + p), \ldots, n - (6 + m_2 + 2p)$, asymmetric $\omega_T^{X11}(B)$ reproduces and asymmetric $\omega_I^{X11}(B)$ annihilates polynomials up to degree 2. For $t = r + p + 1, \ldots, n - (r + p)$, the symmetric filters apply, and symmetric $\omega_T^{X11}(B)$ reproduces and symmetric $\omega_I^{X11}(B)$ annihilates polynomials up to degree 3 (Table 2).

For quarterly series we change 6 to 2 and 7 to 3 in the ranges of time points in Table 4. At the beginning of the series, the first two time point ranges for the quarterly trend and irregular asymmetric filters are then $t = 1, \ldots 2 + m_2 + 2p$ and $t = 3 + m_2 + 2p, \ldots, r + p$. At the end of the series the corresponding time point ranges are $t = n + 1 - (2 + m_2 + 2p), \ldots, n$ and $t = n + 1 - (r + p), \ldots, n - (2 + m_2 + 2p)$. The range where the symmetric quarterly trend and irregular filters apply is still $t = r + p + 1, \ldots, n - (r + p)$.

The presence of just the single $1 - B$ factor in $\omega_S^{X11}(B)$ and $\omega_I^{X11}(B)$ applied near the ends of series means that these filters will under-difference the series unless the appropriate model for the data has just $d = 1$. For the more common case where the model assumes $d = 2$, this fact precludes some calculations one might wish to carry out with original X-11 concurrent filters. Consider, for example, the concurrent seasonal adjustment error, which is $N_n - \omega_N^{X11}(B)y_n = \omega_S^{X11}(B)N_n - \omega_N^{X11}(B)S_n$. This error is nonstationary if $d = 2$, since the concurrent $\omega_S^{X11}(B)$ contains only one difference. Hence, given a model for y_t with $d = 2$, the MSE of a seasonal adjustment using an original X-11 concurrent filter cannot be calculated. For this reason, Bell et al. (2012) made seasonal adjustment MSE calculations, and Bell and Kramer (1999) developed an approach to computing X-11 seasonal adjustment variances, only for X-11 filters with full forecast extension.

When X-11 is applied to a time series partially extended with forecasts and backcasts, that is, with fewer than r forecasts and backcasts (the number required for application of the symmetric seasonal and seasonal adjustment filters to the extended series), we can infer the unit root properties of the resulting implied asymmetric filters from the results given above, including Table 4 and the theorems of Section 2. Assume that the forecasting model is of the form of (3), that is, that the differencing in the model is $\delta(B) = (1 - B)^{d-1}(1 - B^{12}) = (1 - B)^d U(B)$. We first consider $d = 1$. In this case, from Theorem 1, the forecast extension will reproduce constants and from Table 4, the version of $\omega_N^{X11}(B)$ used will reproduce constants at every time point t. It then follows that X-11 seasonal adjustment with the extended series will also reproduce constants. As forecast extension with $d = 1$ will reproduce only constants, not polynomials of higher degree, for $t \leq r$ and $t > n - r$ X-11 seasonal adjustment with the forecast extended series will reproduce only constants.

Consider now the case of $d = 2$ and assume one year of both forecast and backcast extension. (One year forecast extension, but no backcast extension, is the default choice in X-11-ARIMA and X-12-ARIMA.) For simplicity, we will consider what happens at the beginning of the time series. Parallel results hold at the end. Theorem 1 of Section 2 says that, for $d = 2$, forecast and backcast extension will reproduce linear functions of time, but not polynomials of higher degree. With the one-year backcast extension, the seasonal filters that apply at $t = 1, 2, \ldots$ are the asymmetric versions of $\omega_S^{X11}(B)$ that would actually apply at $t = 13, 14, \ldots$ without backcast extension. The results of Table 4 then provide the differencing factors in the asymmetric seasonal filters, but with the time point ranges (at the beginning of the series) shifted by subtracting 12. We thus see the following. (*i*) For $t = 1, \ldots, m_2 + p - 6$, asymmetric $\omega_S^{X11}(B)$ contains $1 - B$, so that seasonal adjustment for these time points reproduces only constants. (*ii*) For $t = m_2 + p - 5, \ldots, r - 12$, asymmetric $\omega_S^{X11}(B)$ contains $(1 - B)^3$, which annihilates quadratics, but since forecasting only reproduces linear functions, seasonal adjustment for these time points reproduces only linear functions. (*iii*) For $t = r - 11, \ldots, r$, the symmetric version of $\omega_S^{X11}(B)$ is applied to the extended series. While it contains $(1 - B)^6$, again the limiting factor is the forecast and backcast extension, so seasonal adjustment for these time points still reproduces only linear functions. Note the corresponding result in Table 3 for the case of $d = 2$. (*iv*) For $t = r + 1, \ldots, n - r$, the symmetric seasonal and seasonal adjustment filters apply using only observed data (no forecast or backcast extension needed), and seasonal adjustment reproduces polynomials up to degree 5.

Similar reasoning can be used to infer properties of the X-11 trend and irregular filters when applied with limited forecast and backcast extension, as well as to infer unit root properties of X-11 filters when applied to series extended with more or fewer forecasts and backcasts. We could also obtain results for values of $d > 2$, but as noted earlier such results would be of little practical relevance.

Finally, it should be noted that the effective half-length of an X-11 symmetric seasonal filter is, in practical terms, much less than the $r = 18 + m_1 + m_2 + p$ on which the results given here are based. This is because the X-11 filter weights are quite small beyond a certain point much less than r. From plots given in Bell and Monsell (1992) of X-11 symmetric filter weights (covering filters generated from the 3×1, 3×3, default 3×5, and 3×9 seasonal MAs, and the 9-, 13-, and 23-term Henderson trend MAs), one might judge that the effective half-length of an X-11 symmetric seasonal filter is about m_2, or perhaps, to be safe, $m_2 + s$ (with $s = 12$ for monthly, and $s = 4$ for quarterly series). Thus, while the asymmetric X-11 seasonal filters exactly include $(1 - B)^6$ only for $t = r + 1, \ldots, n - r$, they may come close to doing so for $t = m_2 + 1, \ldots, n - m_2$, or possibly for $t = m_2 + s + 1, \ldots, n - m_2 - s$. Section 7 now illustrates this point.

7. Illustration

We now illustrate the results of Section 6 on reproduction of polynomials by original X-11 seasonal adjustment and trend filters. For simplicity, quarterly rather than monthly seasonal adjustment was carried out. The input series to X-11 were polynomials of degrees

1 through 5 covering 15 years plus one quarter, or 61 observations. These were of the form $y_t = 30 \times [(t-31)/30]^k$ for $t = 1, \ldots, 61$ and $k = 1, \ldots, 5$. The series values thus ranged from $-30, \ldots, 30$ for odd powers, and from 30 down to 0 and then back up to 30 for even powers. This makes the average absolute quarter-to-quarter change equal to 1 in all cases, so that errors in the seasonal adjustments and trend estimates—the differences between these values and the input polynomial trends—generally reflect errors relative to the average absolute quarter-to-quarter changes.

To keep the X-11 filters relatively short, we specified a 3×3 seasonal MA and a 5-term Henderson trend MA. For these MA choices, $m_1 = m_2 = 2 \times 4 = 8$ and $p = 2$, so $r = 6 + m_1 + m_2 + p = 24$ is the half-length of the X-11 symmetric seasonal adjustment filter, which thus (Table 4) reproduces all the polynomials up to degree 5 for time points $25, \ldots, 37 (= 61-24)$. From Table 4 (with the modifications noted for quarterly series), the X-11 asymmetric seasonal adjustment filters will reproduce polynomials only up to degree 2 at time points $3 + m_2 + p = 13, \ldots, 24 = r$, and similarly at time points $38, \ldots, 49$. At time points $1, \ldots, 12$ and $50, \ldots, 61$, only constants are exactly reproduced by the X-11 seasonal adjustment filters.

The half-length of our X-11 trend filter is $r + p = 26$, so it reproduces polynomials up to degree 3 for time points $27, \ldots, 35$. It reproduces only constant, linear, and quadratic polynomials at time points $15, \ldots, 26$ and $36, \ldots, 47$. At time points $1, \ldots, 14$ and $48, \ldots, 61$, the trend filters reproduce only constants.

Figure 1 displays the results. The seasonal adjustment errors are displayed in the left column of plots for the input polynomials of degrees 1 to 5. The corresponding trend estimation errors are displayed in the right column of plots. The dotted vertical lines in the plots are the limits of the intervals over which the filters reproduce the respective polynomials. No dotted vertical lines appear in the last two plots of the right column, since for these cases none of the input polynomial values are reproduced. Though many of the plotted points outside the region denoted by the dotted vertical lines appear to fall on the horizontal axis, these values are not exactly zero, just too small for their differences from zero to be visually detected on the plots.

For the linear polynomial plots in the first row, we see the magnitude of the errors in both the seasonally adjusted values and trend estimates is quite small, even at the very ends of the series. This shows that while the X-11 asymmetric filters exactly reproduce only constants near the ends of the series, they come very close to reproducing linear polynomials. As we look down the rows of plots, we notice that the magnitude of the errors increases with the degree of the input polynomial, and, especially for the higher degrees, the errors are not so trivial as they were for the linear polynomial. There is generally a seasonal pattern to the errors and, apart from this, the magnitudes of the errors tend to be larger nearer to the ends of the series. The larger errors near the ends of the series are also due to the fact that, for polynomials of degree 2 and higher, the absolute rates of change in the series used increase as one approaches either end of the series. Overall, the largest errors occur for trend estimates at the first and last two time points.

The results in Figure 1 illustrate the point made at the end of Section 6 that the effective half-lengths of the X-11 symmetric filters are considerably less than the exact half-lengths. The plots in the left column suggest that $\omega_N^{X11}(B)$ nearly reproduces polynomials up to degree 5 except for the first and last 10 or so quarters, and $\omega_T^{X11}(B)$ nearly reproduces

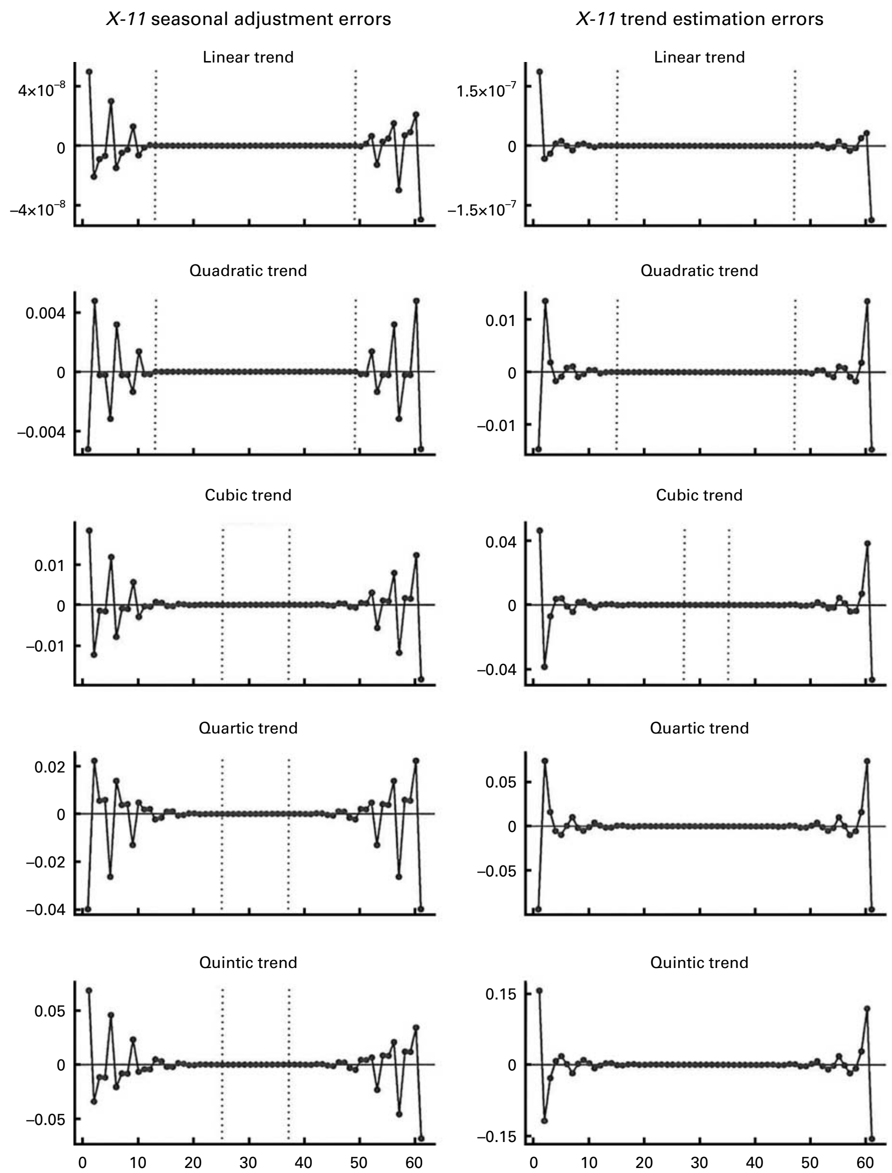

Fig. 1. Seasonal adjustment and trend estimation errors for X-11 quarterly filters with no forecast extension applied to polynomials of degrees one through five. The X-11 filters use 3 × 3 seasonal MAs and a 5-term Henderson trend MA. The polynomials are of the form $y_t = 30 \times [(t-31)/30]^k$ for $t = 1, \ldots, 61$ and $k = 1, \ldots, 5$. The errors are zero within the spans denoted by the dotted vertical lines.

polynomials up to degree 3 except for the first and last seven or so quarters. This contrasts with the ranges over which the corresponding exact results hold, which exclude the first and last 24 and 26 quarters, respectively. Asymmetric $\omega_T^{X11}(B)$ also nearly reproduces polynomials of degrees 4 and 5 apart from the first and last 2–3 years, although even the symmetric version of $\omega_T^{X11}(B)$ does not do so exactly.

8. Seasonal Unit Root Properties of Filters

It is easy to summarize the results on the seasonal unit root properties of the filters considered here. First, all seasonal adjustment filters considered—model-based and X-11, symmetric and asymmetric—contain $U(B)$, and so annihilate fixed seasonal effects. So do the corresponding trend and irregular filters, while the corresponding seasonal filters all reproduce fixed seasonal effects. Symmetric seasonal adjustment, trend, and irregular filters contain additional seasonal unit root factors, specifically:

- symmetric, infinite model-based filters contain $U(B)U(F)$,
- symmetric X-11 filters, both Wallis's and Young's versions (see Section 4), contain $U(B)(1 + F)$, and
- symmetric, finite model-based filters also contain $U(B)(1 + F)$.

Derivation of these results except for the last follows in the same fashion as for the results on $1 - B$ factors given in Sections 3–6. For further details, see Bell (2010). The last result follows since Findley and Martin (2006, p. 29) observe that any finite symmetric filter that includes a $1 + B$ factor must also include a $1 + F$ factor. (A symmetric finite filter is a symmetric filter used to produce estimates at $t = (n + 1)/2$ for n odd.)

Seasonal adjustment filters that include $U(B)U(F)$ annihilate not just fixed seasonal effects, but any deterministic function ξ_t such that $U(B)U(B)\xi_t = 0$. From results on solutions to homogeneous difference equations (Goldberg 1986), such a ξ_t can be shown to be a seasonal pattern whose amplitude grows linearly over time. The models discussed in Section 2 that are used for seasonal adjustment are not really aimed at modeling such increasing amplitude seasonal effects since the models will not, for example, reproduce such patterns in forecasting. The model-based symmetric signal extraction filters reproduce or annihilate such effects simply because, from Equations (8)–(11), these filters include $U(B)$ and also $(1 - B)^d$ as conjugate pairs, that is, as $U(B)U(F)$ and $(1 - B)^d(1 - F)^d$.

The inclusion of $U(B)U(F)$ in model-based symmetric seasonal adjustment, trend, and irregular filters implies that the spectra of the resulting $\hat{N}_t$, $\hat{T}_t$ and $\hat{I}_t$ will have zeros at the seasonal frequencies ($2\pi j/12$ for $j = 1, \ldots, 6$ for monthly series, $\pi/2$ and π for quarterly series). This can be called "overadjustment," a term that refers more generally to dips at the seasonal frequencies (not necessarily to zero) in the spectra of $\hat{N}_t$, $\hat{T}_t$ or $\hat{I}_t$. Evidence of overadjustment (from examination of estimated spectra of estimated components) has long been considered as potentially indicative of problems with the seasonal adjustment. See, for example, Granger (1978). Sims (1978) and Tukey (1978), however, in discussing Granger's article, both pointed out that this was an unrealistic criterion because such "overadjustment" simply follows for model-based adjustment as a consequence of MMSE prediction. In any case, any assessment of the spectral properties of the estimated components from a model-based adjustment should take into account these results. In doing this, note that estimated components from long but finite series will (approximately) show properties of symmetric infinite filtering in the center of the series, but will show properties of asymmetric infinite filtering nearer the ends, where there is no overadjustment. The estimated spectra of $\hat{N}_t$, $\hat{T}_t$ or $\hat{I}_t$ will then show a mixture of these properties. This may produce dips, though not actual zeros, at the seasonal frequencies.

The inclusion of only $U(B)(1+F)$ by the X-11 symmetric seasonal adjustment, trend, and irregular filters, rather than the full $U(B)U(F)$, is a result that no model-based symmetric infinite MMSE filter can produce. However, if one computes transfer functions of the filters $\omega_N^{X11}(B)/U(B)$, where $\omega_N^{X11}(B)$ is the X-11 symmetric seasonal adjustment filter, one finds that these transfer functions very nearly reach 0 at all the seasonal frequencies. (They are exactly 0 at the frequency π due to the symmetric $\omega_N^{X11}(B)$ containing the additional $1+F$ factor, whose zero is at $F=-1=e^{i\pi}$.) This shows that the X-11 symmetric seasonal adjustment filters, and thus the X-11 symmetric trend and irregular filters as well, very nearly include $U(B)U(F)$. This may be partly why Cleveland and Tiao (1976), Burridge and Wallis (1984), and Planas and Depoutot (2002) were successful at finding models whose symmetric infinite filters could well-approximate X-11 symmetric filters.

9. Discussion

We have presented here an essentially complete catalog of results on unit root factors in commonly used seasonal, seasonal adjustment, trend, and irregular linear filters, both model-based and from X-11 with or without forecast extension (full or partial). The unit root factors of interest are differencing operators ($(1-B)^d$ for some $d>0$) and seasonal summation operators ($U(B)=1+B+\ldots+B^{11}$ for monthly data, $U(B)=1+B+B^2+B^3$ for quarterly data), as these determine the extent to which the various filters annihilate or reproduce (i) polynomials in time and (ii) fixed seasonal effects. Differences between the results for various cases were noted. For example, symmetric filters include more (higher order) unit root factors than do the corresponding asymmetric filters.

It is difficult to draw any general conclusions about whether the differences that exist in unit root factors between model-based and X-11 filters favor one or the other, or are generally neutral. Such conclusions, when possible, would presumably depend on the properties of the time series being seasonally adjusted. We can say, however, that while the relation between unit root factors for model-based symmetric and asymmetric filters stems from established statistical principles of MMSE linear projection, the relation between unit root factors for X-11 symmetric and asymmetric filters (with or without forecast extension) is ad hoc. This could raise concerns for some X-11 filters in certain specific instances. On the other hand, results from the illustration of Section 7, such as those showing that X-11 asymmetric seasonal adjustment and trend filters without forecast extension come very close to reproducing linear polynomials, though they exactly reproduce only constants, means we must be cautious in how we interpret the exact results.

This last remark also reminds us that the exact results presented should nonetheless hold approximately in other settings where one filter well-approximates another. For example, the results on unit roots of model-based symmetric filters strictly apply only to the case of seasonal adjustment using a doubly infinite realization of a time series ($\{y_t$ for $t=-\infty,\ldots,\infty\}$), a situation never exactly realized in practice. However, model-based filter weights from the models considered here die out with increasing lead or lag, so that in the middle of a sufficiently long series the symmetric filters nearly apply, and then we can expect the unit root results for symmetric filters to hold approximately. Similarly, the ends of X-11 symmetric filters contain a large number of very small (in magnitude) weights, so that the effective length of the filters is considerably less than their exact length, and the results on unit roots in X-11 symmetric filters will apply approximately over much wider time intervals than those over which the results apply exactly. This can be seen in the results of Section 7. Precisely how long a series needs to be in order to be considered "sufficiently long" for the filters being applied at specific time points to be regarded as approximately symmetric, or how many forecasts are really needed to approximate "full forecast extension," will depend on the particular fitted models and filters being used, so this must be judged on a case-by-case basis.

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