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Using a Weighted Average of Base Period Price Indexes to Approximate a Superlative Index

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The Lloyd–Moulton price index has been advocated as a timely approximation to a superlative price index. We show that a weighted average of the arithmetic and geometric base-weighted (Laspeyres) indexes can serve as a simple, robust alternative to the Lloyd–Moulton. The parameter needed for the weighted average can be readily and systematically estimated from past data and continuously updated as new data become available. Previous methods of estimating this parameter have entailed either a trial-and-error process, requiring human judgment, or the use of iterative numeric algorithms. An empirical study indicates that we may compute timely, close approximations to a superlative index using a weighted average of the arithmetic and geometric Laspeyres indexes with parameters estimated and systematically updated from prior data.

Key words: Taylor series; elasticity of substitution; Lloyd–Moulton Index; sample survey.

1. Introduction

A *consumer price index* (CPI) is a measure of change from one time period to another of the purchasing power of a given population's monetary unit. A *cost of living index* (COLI) is the ratio of minimal costs needed in the two time periods to achieve a given standard of living. A body of theory suggests that certain "superlative" index formulas give a good approximation to a COLI (Diewert 1987). These formulas have been difficult to implement, however, because they require information on consumer expenditure patterns for both of the two reference periods; such information on the more recent period is usually unavailable at the time of index production.

The desirability of a COLI is not universally accepted. On grounds of simplicity and transparency, many countries prefer a market basket approach, often choosing the Laspeyres-type index described below (U.S. National Research Council 2002, pp. 43–44). Other approaches are discussed in the International Labour Office's Consumer Price Index Manual (2004). In this article however, we present a ready, simple estimation method for applications in which a timely superlative price index is desired.

Shapiro and Wilcox (1997) advocated the Lloyd–Moulton price index (Lloyd 1975; Moulton 1996) as a timely approximation to a superlative index. Instead of second

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period expenditure information, the Lloyd–Moulton relies on a parameter representing "elasticity of substitution" and uses past years' data as a basis for evaluating the parameter estimate. The elasticity of substitution indicates the extent to which consumers change their buying patterns in response to changes in relative prices. Higher elasticity values indicate greater willingness, on the part of consumers, to substitute cheaper items for more expensive ones. Balk (2000) provides numerical methods of estimating the elasticity parameter.

An alternative approach to estimating a COLI, providing greater operational simplicity and flexibility, is to compute a weighted average of the base-weighted arithmetic and geometric (Laspeyres) indexes defined below. We show through Taylor series expansions that the arithmetic-geometric average (or "AG Mean") index closely approximates the Lloyd–Moulton and hence the superlative indexes. The weight applied to the geometric index in the AG Mean may be estimated from prior data through a simple formula and then systematically updated with more recent consumer expenditure data. By contrast, Shapiro and Wilcox (1997) used a trial-and-error method to approximate a parameter for the Lloyd–Moulton index, and the parameter was then held constant throughout the time period studied. Because of the systematic updating, the AG Mean continuously picks up changes in consumer buying patterns reflected in the data. The systematic updating of the AG Mean requires no iterative numerical procedures and can therefore be easily programmed and automated in a statistical production setting. To illustrate the practicality of this approach, we present findings from an empirical study.

2. Price Index Formulas and Estimators

The classic price index formula is the (arithmetic) *Laspeyres* index,

$$L = \frac{\sum_{j=1}^{N} q_{j1} p_{j2}}{\sum_{j=1}^{N} q_{j1} p_{j1}} = \sum_{j=1}^{N} w_{j1} \left(\frac{p_{j2}}{p_{j1}} \right)$$

where p_{jt} denotes the price of item j at time t, q_{jt} denotes the quantity of item j purchased at time t, $w_{jt} = p_{jt} q_{jt} / \sum_{k=1}^{N} p_{kt} q_{kt}$, and N denotes the number of items in the target population. The weight w_{jt} is the *expenditure share* for item j in period $t \in \{1,2\}$; the ratios p_{j2}/p_{j1} are referred to as *price relatives*. Grounded in the "fixed market basket" concept, L is the ratio of the total costs, in the two reference periods, of the bundle of goods and services that were purchased in Period 1. The Laspeyres is similar in principle to the *Paasche* index, given by

$$P = \frac{\sum_{j=1}^{N} q_{j2} p_{j2}}{\sum_{j=1}^{N} q_{j2} p_{j1}} = \left[\sum_{j=1}^{N} w_{j2} \left(p_{j2}/p_{j1} \right)^{-1} \right]^{-1}$$

which is based on quantities for Period 2. Estimation of the Laspeyres index is more practical, because estimates of the first period shares w_{j1} are more likely to be available at Period 2 than are estimates of w_{j2}, which are based on Period 2 expenditure share weights. On the basis of some postulated desirable properties of price indexes, Irving Fisher (1922) suggested that the *ideal* index would be $F = \sqrt{LP}$, which has come to be called the *Fisher* index.

Another formula that now plays an important role in the U.S. CPI is a *geometric mean* of the price relatives,

$$G = \exp\left\{\sum_{j=1}^{N} w_j \ln\left(\frac{p_{j2}}{p_{j1}}\right)\right\} = \prod_{j=1}^{N}\left(\frac{p_{j2}}{p_{j1}}\right)^{w_j}$$

which is a generalization of the unweighted geometric mean known as the *Jevons* index, after its originator. The weights w_j in G might be fixed across time or be taken as the first or second period shares defined above. When first period weights are used, the index G is sometimes referred to as the geometric Laspeyres index. The *Törnqvist* index is a geometric mean index with weights based on the arithmetic average of the expenditure share weights across the two reference periods, i.e.,

$$T = \prod_{j=1}^{N}\left(\frac{p_{j2}}{p_{j1}}\right)^{w_{j,1,2}}$$

where $w_{j,1,2} = (w_{j1} + w_{j2})/2$. Both the Fisher and Törnqvist indexes are known as *superlative* indexes, because economic theory suggests that, under relatively weak assumptions, they approximate a COLI (Diewert 1987).

In practice, government agencies apply a given price index formula to a *sample* from the target population, yielding an *index estimator* of the selected population index. In most cases, neither estimated quantities nor expenditure share weights are available for either of the price index reference Periods 1 and 2. Statistical agencies often use expenditure share weights estimated for some earlier Period 0. Combining these weights with price data from Periods 1 and 2, they may estimate a modified Laspeyres index or weighted geometric mean index as

$$\hat{L}_{0,1,2} = \sum_{j=1}^{n} \hat{w}_{j0}\left(\frac{p_{j2}}{p_{j1}}\right)$$

or

$$\hat{G}_{0,1,2} = \prod_{j=1}^{n}\left(\frac{p_{j2}}{p_{j1}}\right)^{\hat{w}_{j0}}$$

respectively, where n is the sample size and $\hat{w}_{j0}$ is an estimated expenditure share for item j in Period 0. In what follows, we put aside this complication.

For sampling purposes, government agencies often categorize the population of consumer items into groups defined by item characteristics and/or geographic areas and draw a sample of items within each group. This categorization gives rise to "composite forms" of price index formulas. For example, the Laspeyres index (with Period 1 weights) can be written as $L = \sum_g w_{g1} L_g$, where

$$w_{gt} = \frac{\sum_{i=1}^{N_g} q_{git} p_{git}}{\sum_g \sum_{i=1}^{N_g} q_{git} p_{git}}$$

is the expenditure share for the gth group, and L_g is the Laspeyres *sub-index* for group g. The computation of subindexes is called "lower-level aggregation," while the process of combining subindexes into an overall index, often by a different formula, is called "upper-level aggregation." In the empirical study we present in Section 4, we focus on an application of the AG Mean at the upper level of aggregation.

3. The AG Mean Approximations to the Lloyd–Moulton

For general notation, we write $I = X_Y$, where X and Y are the formulas used for upper- and lower-level aggregation, respectively. A composite price index estimator $\hat{I}(\hat{w}_g, \hat{I}_g)$ combines subindexes $\hat{I}_g$ with expenditure weights $\hat{w}_g$, e.g., $\hat{w}_g \in \{\hat{w}_{g1}, \hat{w}_{g2}\}$, where $\hat{w}_{gt}$ denotes the estimated expenditure share at time t for a stratum g. Thus we have the base (Period 1) weighted geometric mean index $\hat{G}_{\hat{I}} = \prod_g \hat{I}_g^{\hat{w}_{g1}}$ and the base-weighted Laspeyres index $\hat{L}_{\hat{I}} = \sum \hat{w}_{g1} \hat{I}_g$.

The Lloyd–Moulton (or CES) index estimator is defined as

$$\hat{C}_{\hat{I}} = \left\{ \sum_g \hat{w}_{g1} \hat{I}_g^{1-\tau} \right\}^{1/(1-\tau)}$$

and has been shown to target a population COLI, as approximated by a Törnqvist or Fisher index. The parameter τ is called the "elasticity of substitution." Note that $\hat{C}_{\hat{I}} \rightarrow \hat{G}_{\hat{I}}$ as $\tau \rightarrow 1$. (Although the elasticity parameter is often denoted by σ, we use τ here, because σ will be used for a different purpose below.)

We may approximate $\hat{C}_{\hat{I}}$ by a weighted arithmetic or geometric average of the base-weighted geometric mean $(\hat{G}_{\hat{I}})$ and Laspeyres indexes, with τ as the weight assigned to $\hat{G}_{\hat{I}}$. Let

$$\tilde{C}_{\hat{I},\alpha} = \tau \hat{G}_{\hat{I}} + (1 - \tau) \hat{L}_{\hat{I}}$$

be the arithmetic *AG Mean* index.

To see that $\tilde{C}_{\hat{I},\alpha}$ is an appropriate estimator of $\hat{C}_{\hat{I}}$, we expand $\hat{C}_{\hat{I}}$ about a constant vector, using a generalization of the approach suggested by Dalén (1992). (Other approaches to the use of Taylor series expansions to compare price indexes differ slightly from Dalén's in both method and purpose; see, for example, Diewert 1987.) For notational simplicity, let $w_g = \hat{w}_{g1}$, and let

$$\mu_\alpha = \sum_g w_g \hat{I}_g,\ \sigma_\alpha^2 = \sum_g w_g (\hat{I}_g - \mu_\alpha)^2, \text{ and } \gamma_\alpha = \sum_g w_g (\hat{I}_g - \mu_\alpha)^3$$

When we approximate $\hat{C}_{\hat{I}}$ by a Taylor series expansion about the point $\hat{I}_g = \mu_\alpha$ for all g, and express the third-order expansion in terms of the above moments, we have

$$\hat{C}_{\hat{I}} \approx \mu_\alpha - \frac{\tau\sigma_\alpha^2}{2\mu_\alpha} + \frac{\tau(\tau+1)\gamma_\alpha}{6\mu_\alpha^2} \tag{1}$$

(The relevant derivatives are given in the Appendix.) Note that $\mu_\alpha = \hat{L}_{\hat{I}}$. Setting $\tau = 1$, we obtain the third-order expansion of the base-weighted Geometric mean index $\hat{G}_{\hat{I}} = \prod_g \hat{I}_g^{w_g}$:

$$\hat{G}_{\hat{I}} \approx \mu_\alpha - \frac{\sigma_\alpha^2}{2\mu_\alpha} + \frac{\gamma_\alpha}{3\mu_\alpha^2} \tag{2}$$

So from (1), we have, to third order,

$$\hat{C}_{\hat{I},\alpha} \approx \mu_\alpha - \frac{\tau\sigma_\alpha^2}{2\mu_\alpha} + \frac{\tau\gamma_\alpha}{3\mu_\alpha^2}$$

so

$$\tilde{C}_{\hat{I},\alpha} - \hat{C}_{\hat{I}} \approx \frac{\tau(1-\tau)\gamma_\alpha}{6\mu_\alpha^2}$$

With $\tau \in [0,1]$,

$$\frac{\tau(1-\tau)\gamma_\alpha}{6\mu_\alpha^2} \leq \frac{\gamma_\alpha}{24\mu_\alpha^2}$$

so the difference $\tilde{C}_{\hat{I},\alpha} - \hat{C}_{\hat{I}}$ should be small provided γ_α is reasonably small. As the empirical results in the next section illustrate, this third order difference is often negligible.

When $\tilde{C}_{\hat{I},\alpha}$ is used to approximate the Fisher index $\hat{F}_{\hat{I}}$, the appropriate weight of $\hat{G}_{\hat{I}}$ in the average is simply

$$\theta_{F,\alpha} = \frac{\hat{L}_{\hat{I}} - \hat{F}_{\hat{I}}}{\hat{L}_{\hat{I}} - \hat{G}_{\hat{I}}} \tag{3}$$

obtained by setting $\tilde{C}_{\hat{I},\alpha}$ equal to $\hat{F}_{\hat{I}}$ with $\tau = \theta_{F,\alpha}$ and solving for $\theta_{F,\alpha}$. Similarly, $\theta_{T,\alpha} = (\hat{L}_{\hat{I}} - \hat{T}_{\hat{I}})/(\hat{L}_{\hat{I}} - \hat{G}_{\hat{I}})$ is appropriate when the estimation target is a Törnqvist index. Note that $\theta_{F,\alpha}$ and $\theta_{T,\alpha}$ rely on the index estimates $\hat{F}_{\hat{I}}$ and $\hat{T}_{\hat{I}}$, respectively. Because the elasticity of substitution does not normally change rapidly over time, however, we can use the index estimates calculated from prior data to obtain current values of $\theta_{F,\alpha}$ and $\theta_{T,\alpha}$. These parameter estimates can then be continuously and systematically updated. The closed algebraic forms of $\theta_{F,\alpha}$ and $\theta_{T,\alpha}$ are convenient for analysis purposes and can be easily programmed, e.g., as part of a data processing system used for scheduled production computations.

We also consider a geometric AG Mean, $\tilde{C}_{\hat{I},\xi} = \hat{G}_{\hat{I}}^{\tau}\hat{L}_{\hat{I}}^{1-\tau}$, which can also be used as an alternative to the Lloyd–Moulton. Törnqvist (1936, quoted by Vartia 1978) provides the following third-order expansion of log $\left(\hat{C}_{\hat{I}}\right)$:

$$\log\left(\hat{C}_{\hat{I}}\right) \approx \mu_{\xi} + \frac{(1-\tau)\sigma_{\xi}^{2}}{2} + \frac{(1-\tau)^{2}\gamma_{\xi}}{6} \tag{4}$$

where

$$\mu_{\xi} = \sum_{g} w_{g} \log\left(\hat{I}_{g}\right),\ \sigma_{\xi}^{2} = \sum_{g} w_{g}\left[\log\left(\hat{I}_{g}\right) - \mu_{\xi}\right]^{2}$$

and

$$\gamma_{\xi} = \sum_{g} w_{g}\left[\log\left(\hat{I}_{g}\right) - \mu_{\xi}\right]^{3}$$

Setting $\tau = 0$ in (4), we have

$$\log\left(\hat{L}_{\hat{I}}\right) \approx \mu_{\xi} + \frac{\sigma_{\xi}^{2}}{2} + \frac{\gamma_{\xi}}{6}$$

So we may write

$$\tau \log\left(\hat{G}_{\hat{I}}\right) + (1-\tau)\log\left(\hat{L}_{\hat{I}}\right) \approx \mu_{\xi} + \frac{(1-\tau)\sigma_{\xi}^{2}}{2} + \frac{(1-\tau)\gamma_{\xi}}{6} \tag{5}$$

(since log $\left(\hat{G}_{\hat{I}}\right) = \mu_{\xi}$). From (4) and (5), we have, to third-order

$$\log\left(\tilde{C}_{\hat{I},\xi}\right) - \log\left(\hat{C}_{\hat{I}}\right) \approx \frac{\tau(1-\tau)\gamma_{\xi}}{6} \leq \frac{\gamma_{\xi}}{24} \tag{6}$$

when $\tau \in [0,1]$. Thus $\tilde{C}_{\hat{I},\xi}$ should also serve as a good approximation to $\hat{C}_{\hat{I}}$. To estimate the parameter τ for $\tilde{C}_{\hat{I},\xi}$, we may use

$$\theta_{T,\xi} = \left[\log\left(\hat{L}_{\hat{I}}\right) - \log\left(\hat{T}_{\hat{I}}\right)\right]\left[\log\left(\hat{L}_{\hat{I}}\right) - \log\left(\hat{G}_{\hat{I}}\right)\right]^{-1}$$

or

$$\theta_{F,\xi} = \left[\log\left(\hat{L}_{\hat{I}}\right) - \log\left(\hat{F}_{\hat{I}}\right)\right]\left[\log\left(\hat{L}_{\hat{I}}\right) - \log\left(\hat{G}_{\hat{I}}\right)\right]^{-1}$$

which are analogous to (3) above.

Using our estimates of τ from (3) or (6), we could calculate a Lloyd–Moulton index, $\hat{C}_{\hat{I}}$, but $\tilde{C}_{\hat{I},\alpha}$ and $\tilde{C}_{\hat{I},\xi}$ are computationally simpler. In the case $\tau = 1$, all three are equivalent to $\hat{G}_{\hat{I}}$, but $\hat{C}_{\hat{I}}$ cannot be directly computed. For very large values of τ (approaching infinity), $\tilde{C}_{\hat{I},\alpha}$ and $\tilde{C}_{\hat{I},\xi}$ may not be good approximations, but very large values of τ are unlikely to be appropriate for consumer price indexes.

4. Empirical Results

To test the usefulness of the AG Mean, we performed an empirical study using airfare data from the Passenger Origin and Destination (O&D) Survey, a quarterly survey conducted by the U.S. Bureau of Transportation Statistics (BTS). Our data comprised unit value subindexes $\hat{I}_{g}$ (average price in time t divided by average price in time $t-1$) for detailed categories of airline itineraries, along with corresponding expenditure share weights w_{g}.

Both the $\hat{I}_g$and the w_g were computed from the O&D survey data, at the most detailed level of aggregation allowed by the data. Each group g comprises itineraries flown during the reference quarter on the same sequence of carriers with identical trip routes. The dates and flight times for the itineraries may differ within a group g, but the trip route, sequence of class of service categories (e.g., coach, first class), and sequence of air carriers are the same for all itineraries within a group. (For details on the survey and the estimation method, see Lent and Dorfman 2005. BTS is now publishing the Air Travel Price Index series, computed by this method, as a quarterly research series.) We computed superlative index estimates using the formula $\hat{F}_{\hat{I}}$. We then estimated elasticity parameters based on Formula (3) above and applied the AG Mean approximations. Our results illustrate the use of the AG Mean, with systematic parameter updates, in the case of chained index series.

The figures below show comparisons between the quarterly chained Fisher, Laspeyres, and arithmetic AG Mean index series for the period between the 4th quarter of 1998 and the 3rd quarter of 2003. In this application, the differences between the Laspeyres and Fisher index series are, to some extent, due to chain drift in the Laspeyres series (see Lent 2003). We computed the AG Mean indexes, $\tilde{C}_{\hat{I},\alpha}$, using two different estimators of the elasticity parameter $\theta_{F,\alpha}$ both based on four-quarter moving average estimates of parameters computed from previous quarters. The unsmoothed quarterly parameter estimates are given in Table 1. The series labeled "AG Mean 1" was computed assuming a one-quarter lag between the availability of price data and expenditure share weights, while "AG Mean 2" was computed assuming a two-quarter lag. We use lagged data, even though contemporary expenditure estimates are available for the airline data, because, typically in

Table 1. Unsmoothed quarterly elasticity estimates $\theta_{F,\alpha}$

Quarter	New York	Denver	Manchester	San Juan
1998 Q2	0.023	0.079	0.066	0.275
1998 Q3	0.026	0.084	0.055	0.151
1998 Q4	−0.008	0.099	0.024	0.200
1999 Q1	0.134	0.116	0.221	0.147
1999 Q2	0.026	0.079	0.149	0.297
1999 Q3	0.057	0.064	−0.013	0.261
1999 Q4	0.038	0.049	0.039	0.137
2000 Q1	0.184	0.112	0.196	0.105
2000 Q2	0.004	0.086	0.104	0.178
2000 Q3	0.015	0.035	−0.002	0.114
2000 Q4	0.009	0.046	0.008	0.162
2001 Q1	0.112	0.054	0.106	0.029
2001 Q2	0.091	0.125	0.155	0.069
2001 Q3	0.082	0.065	0.088	0.090
2001 Q4	−0.006	0.081	0.037	0.204
2002 Q1	0.138	0.164	0.114	0.221
2002 Q2	0.039	0.155	0.079	0.298
2002 Q3	0.110	0.065	0.005	0.250
2002 Q4	0.055	0.112	0.084	0.260
2003 Q1	0.173	0.230	0.114	0.085
2003 Q2	0.086	0.160	0.082	0.061
2003 Q3	0.049	0.047	−0.020	0.051

practice, only lagged expenditure data are available. For $k \in \{1,2\}$, the elasticity parameter for AG Mean k is

$$\bar{\theta}^{(mk)}_{F,\alpha,t} = \frac{\sum_{l=t-k-3}^{t-k} \theta_{F,\alpha,l}}{4}$$

where $\theta_{F,\alpha,l}$ is the parameter estimate computed using data from quarters $l-1$ and l, and $\bar{\theta}^{(mk)}_{F,\alpha,t}$ is the moving average parameter used in the AG Mean index measuring change between quarters $t-1$ and t. The parameters $\theta_{F,\alpha,l}$ measure the elasticity of substitution between itineraries with different destinations as well as between itineraries with flights operated by different air carriers. The parameters generally run between 0 and 0.10, indicating little substitution between itinerary destinations.

Figure 1 shows the four index series for itineraries originating in New York City or Newark (John F. Kennedy, LaGuardia, or Newark Liberty International Airport). We see very little difference between AG Mean 1 and AG Mean 2, indicating only very gradual change in the moving average elasticities over the five-year period. The AG Mean series do rise very slightly above the Fisher during the later quarters of the series, due to the cumulative effect of a gradual rise in the $\theta_{F,\alpha,l}$ elasticity estimates. These parameters increase from roughly 0.05 to 0.09 during this period, which was marked by the expansion of "low-cost" air carriers and increased competition in the air travel service market. In spite of the low elasticities and the gradual change in the elasticity estimates over the period, both of the AG Mean series provide much better approximations to the Fisher index than does the Laspeyres.

Figures 2 and 3 provide results for itineraries originating in Denver, Colorado and Manchester, New Hampshire (a much smaller market), respectively. In general, the Denver and Manchester series are similar to those for the New York City area. Beginning in 2002, with the expansion of low-cost carrier service from the Denver airport, the elasticity parameter for the Denver AG Mean series increases, causing the AG Mean indexes to run slightly above the Fisher, though well below the Laspeyres. For itineraries originating in Manchester, both of the AG Mean series closely approximate the Fisher. The AG Mean series for the San Juan market, shown in Figure 4, run slightly below the Fisher, due to decreases in the elasticity parameters, which were not seen in the data for any of several U.S. cities examined. The elasticities for San Juan were initially higher than

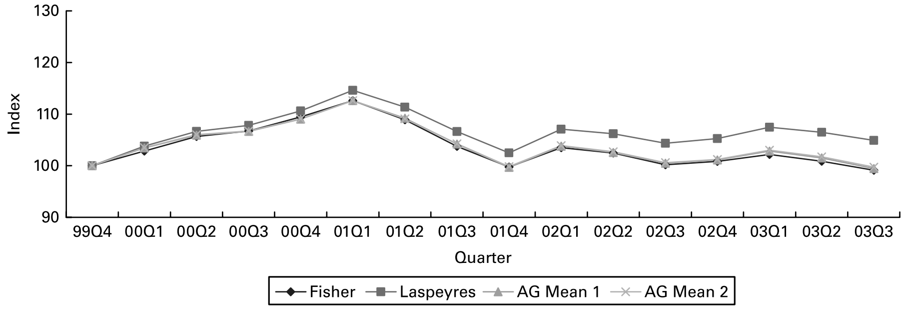

Fig. 1. Alternative airfare indexes for New York, New York and Newark, New Jersey 99Q4 = 100

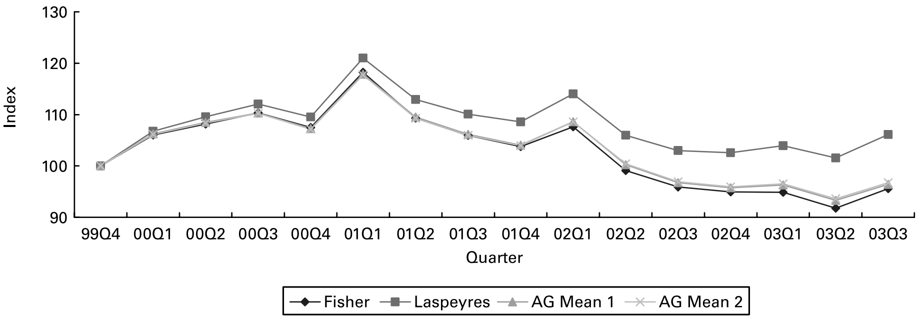

Fig. 2. Alternative airfare indexes for Denver, Colorado 99Q4 = 100

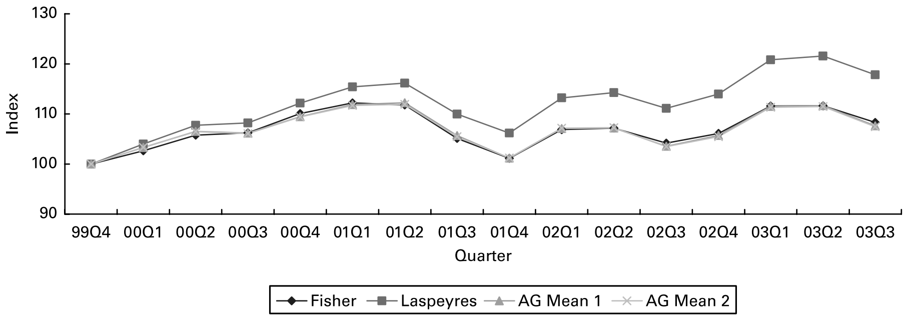

Fig. 3. Alternative airfare indexes for Manchester, New Hampshire 99Q4 = 100

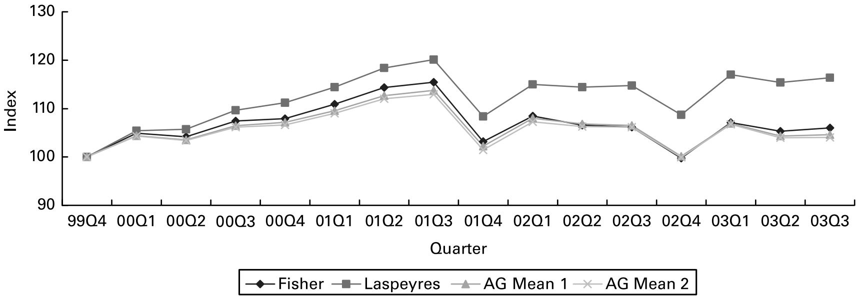

Fig. 4. Alternative airfare indexes San Juan, Puerto Rico 99Q4 = 100

those for the U.S. cities, however, generally running between 0.06 and 0.20 over the test period. In this case also, both of the AG Mean series dramatically outperform the Laspeyres by correcting–albeit with a lag–for the elasticity of substitution between air travel itineraries and carriers.

5. Conclusion

The weighted arithmetic or geometric of the base-weighted Geometric mean and Laspeyres indexes, with simple, continuously updated estimates of elasticity based on lagged data, provides a good on-time approximation to a superlative index. Like the

Lloyd–Moulton index, the AG Mean relies on a parameter that is intuitively meaningful and useful in its own right.

Other methods of using data from prior time periods to approximate a superlative index, e.g., calculating a simple ratio adjustment factor of the Fisher to the Laspeyres index, lack this property. Unlike the Lloyd–Moulton parameter, however, the AG Mean parameter has a closed algebraic form. It can therefore be more easily analyzed, and its update computations can be readily automated for use in large-scale data processing and statistical production systems.

Appendix

To derive expression (1), we expand the function $\hat{C}_{\hat{I}}$ around the point $\hat{\mathbf{I}} = \boldsymbol{\mu} = (\mu, \ldots, \mu)$, where $\mu = \mu_\alpha$. The general formula for the Taylor expansion is

$$f(\hat{\mathbf{I}}) = f(\boldsymbol{\mu}) + \sum f'_g(\boldsymbol{\mu})(\hat{I}_g - \mu) + \frac{1}{2}\sum\sum f''_{g1,g2}(\boldsymbol{\mu})(\hat{I}_{g1} - \mu)(\hat{I}_{g2} - \mu)$$
$$+ \frac{1}{6}\sum\sum\sum f'''_{g1,g2,g3}(\boldsymbol{\mu})(\hat{I}_{g1} - \mu)(\hat{I}_{g2} - \mu)(\hat{I}_{g3} - \mu)$$

The necessary derivatives of $\hat{C}_{\hat{I}}$, evaluated at $\hat{\mathbf{I}} = \boldsymbol{\mu}$ are as follows.

$$\left.\frac{\partial \hat{C}_{\hat{I}}}{\partial \hat{I}_g}\right|_{\hat{\mathbf{I}}=\boldsymbol{\mu}} = w_g$$

$$\left.\frac{\partial^2 \hat{C}_{\hat{I}}}{\partial \hat{I}_g^2}\right|_{\hat{\mathbf{I}}=\boldsymbol{\mu}} = \mu^{-1}\tau w_g(w_g - 1)$$

$$\left.\frac{\partial^2 \hat{C}_{\hat{I}}}{\partial \hat{I}_{g1}\partial \hat{I}_{g2}}\right|_{\hat{\mathbf{I}}=\boldsymbol{\mu}} = \mu^{-1}\tau w_{g1} w_{g2}$$

$$\left.\frac{\partial^3 \hat{C}_{\hat{I}}}{\partial \hat{I}_g^3}\right|_{\hat{\mathbf{I}}=\boldsymbol{\mu}} = \mu^{-2}\left[\tau(1 - 2\tau)w_g^3 - 3\tau^2 w_g^2 + \tau(\tau + 1)w_g\right]$$

$$\left.\frac{\partial^3 \hat{C}_{\hat{I}}}{\partial \hat{I}_{g1}^2 \hat{I}_{g2}}\right|_{\hat{\mathbf{I}}=\boldsymbol{\mu}} = \mu^{-2}\tau\left[(1 - 2\tau)w_{g1}^2 w_{g2} + \tau w_{g1} w_{g2}\right]$$

$$\left.\frac{\partial^3 \hat{C}_{\hat{I}}}{\partial \hat{I}_{g1}\hat{I}_{g2}\hat{I}_{g3}}\right|_{\hat{\mathbf{I}}=\boldsymbol{\mu}} = \mu^{-2}\tau(1 - 2\tau)w_{g1} w_{g2} w_{g3}$$

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